Blocking Semiovals and their Applications in Cryptography

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(joint work with Dr. Jeremy Dover)

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2-dimensional subspaces $\leftrightarrow$ lines

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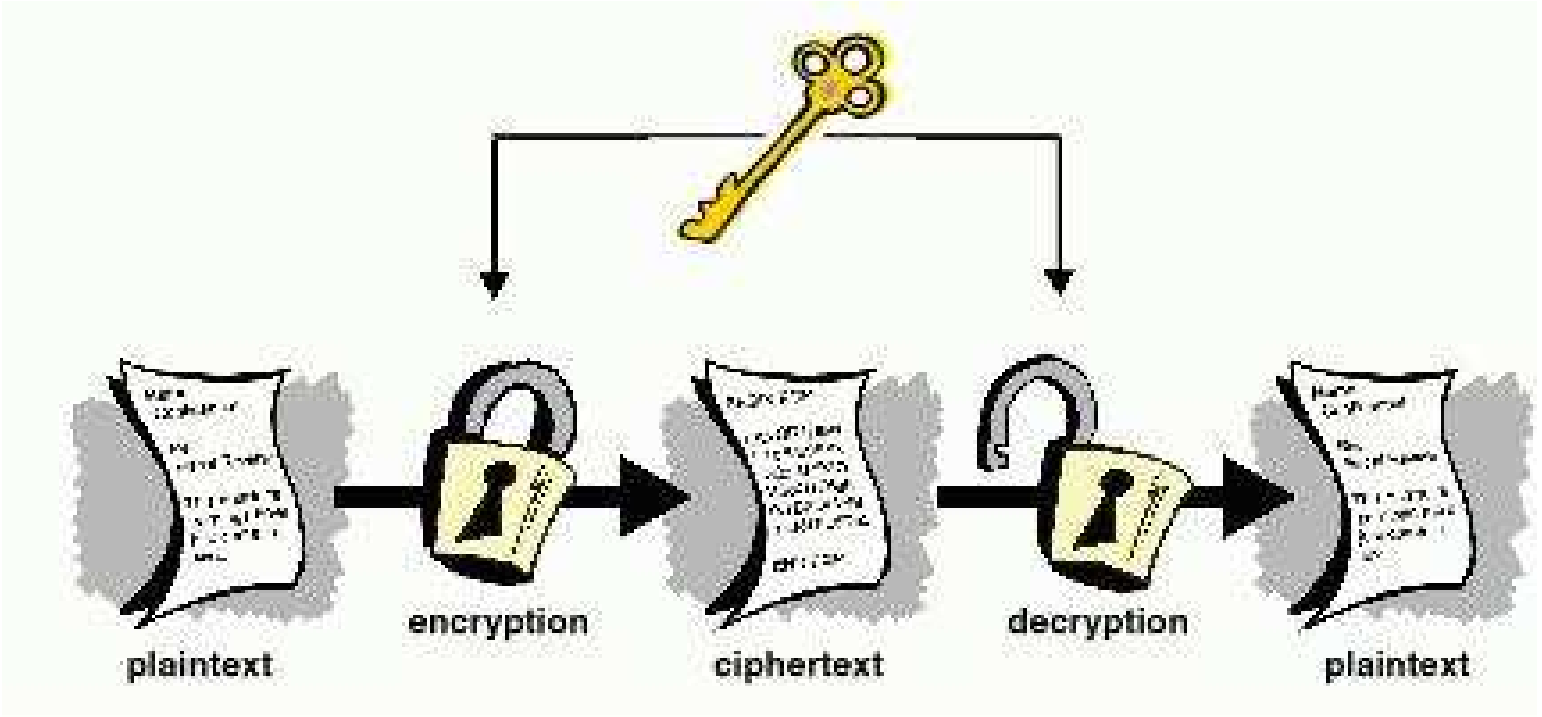
1. Introduce you to a not-so-well-known cryptographic protocol
2. Describe my work in finite geometry that's related

Cryptography



from points/lines to matrices

$$\begin{array}{c} \\ l_1 \\ l_2 \\ l_3 \\ \vdots \\ l_n \end{array} \begin{array}{c} \begin{array}{ccccc} P_1 & P_2 & P_3 & \dots & P_n \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & \dots & 1 \\ 0 & 1 & 0 & \dots & 1 \\ 1 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 1 & \dots & 0 \end{pmatrix} \end{array}$$

ciphertexts as vectors

A cryptographic protocol...

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determining sets

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However, we must avoid passant lines (lines not meeting our set at all), and.. multiple tangent lines.

That is, any particular point of a determining set $\mathcal{S}$ can have at most one tangent line through it.

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3. a cipher ρ that acts on the columns of M_B (the columns corresponding to points of B)
4. a cipher σ that acts on the columns of $M_{\pi \setminus B}$ (the columns *not* corresponding to points of B)

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The cipher σ is private (known only to Alice)

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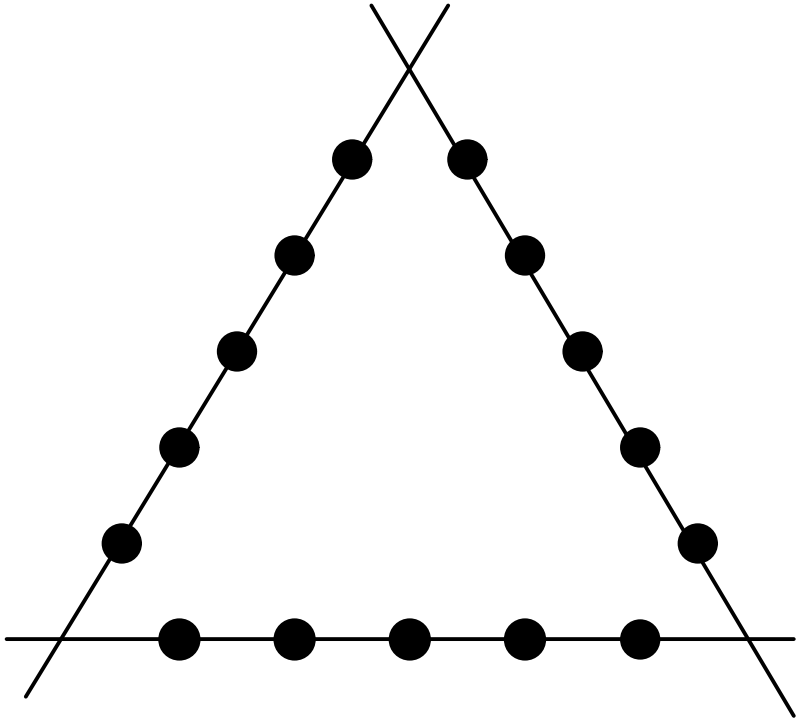
Naturally, a *blocking semioval* is a set that is both a blocking set *and* a semioval.

vertex-less triangle

Classic example: *the vertex-less triangle*



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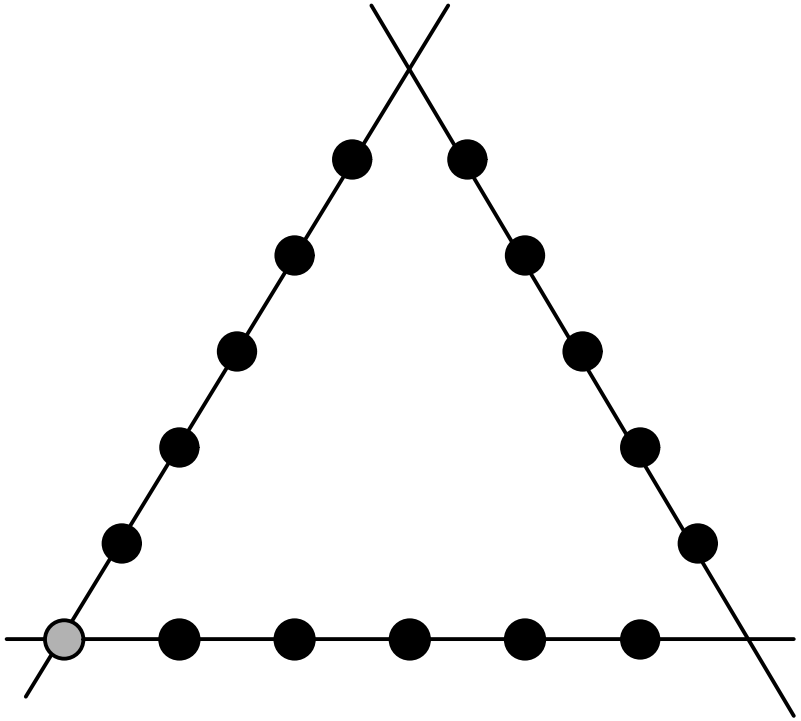

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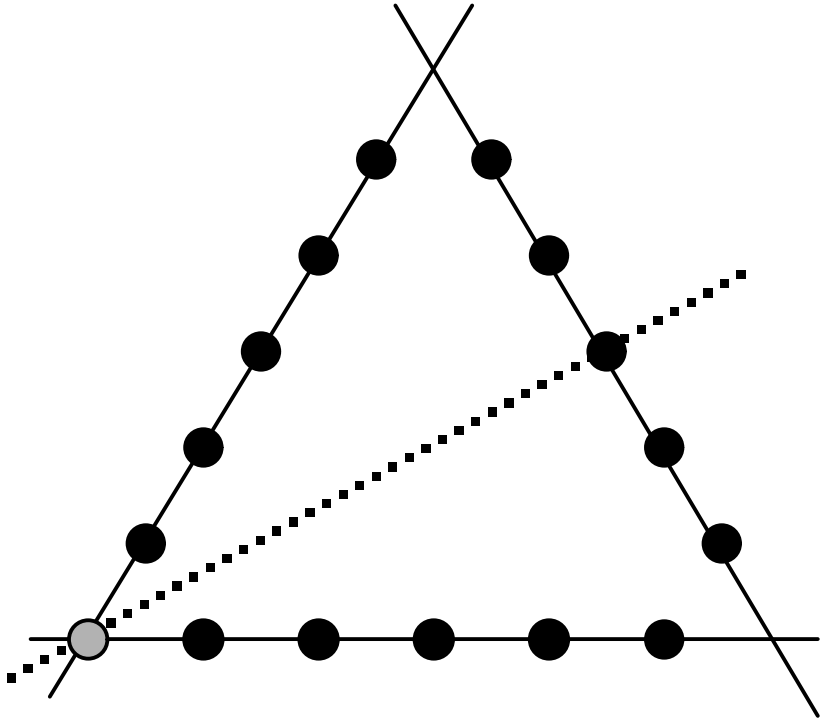

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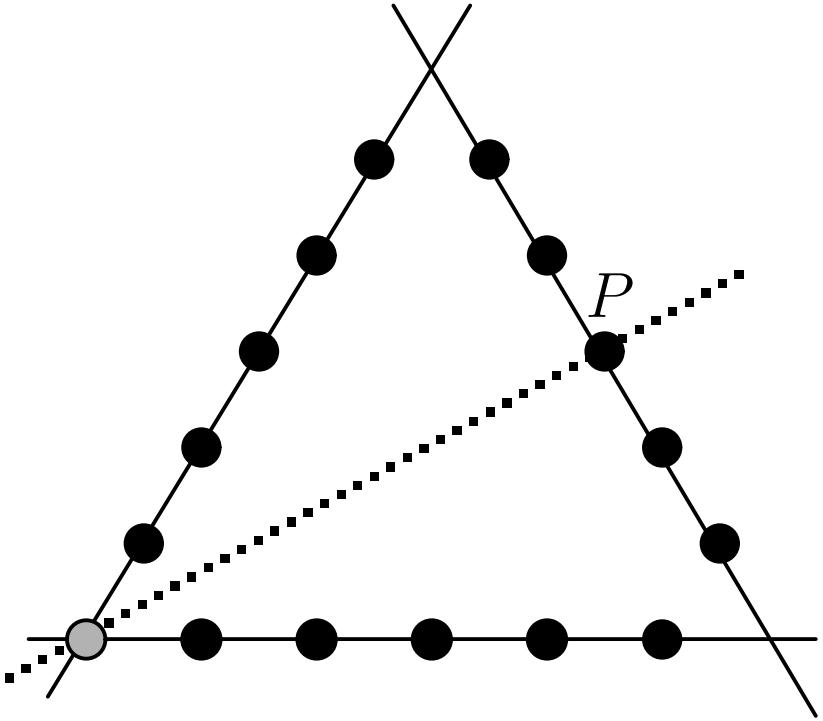

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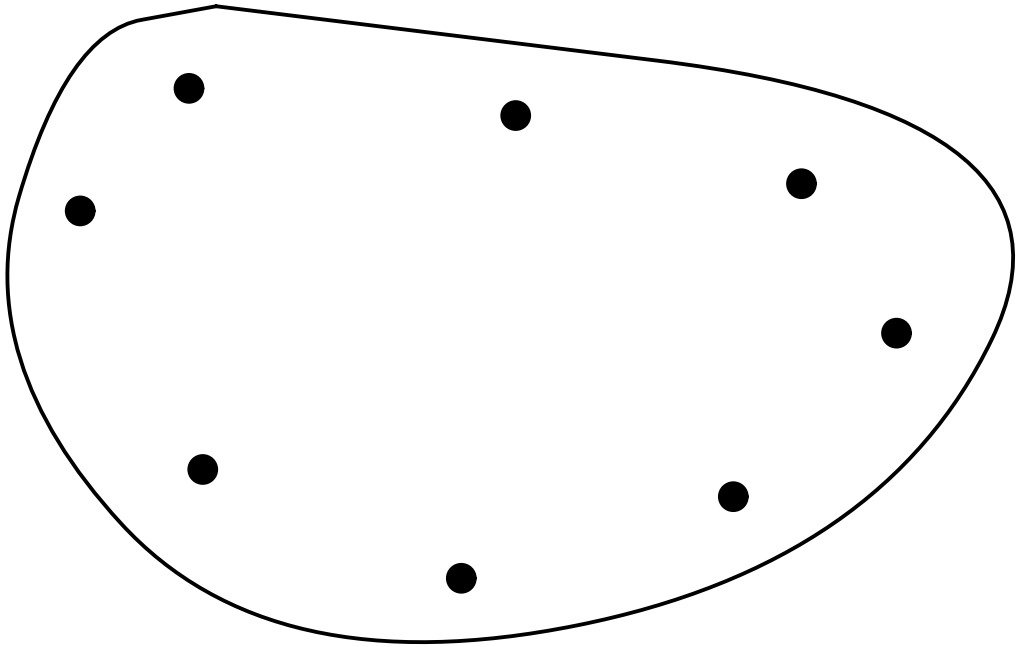
First, you can't remove any point and still have a blocking semioval.



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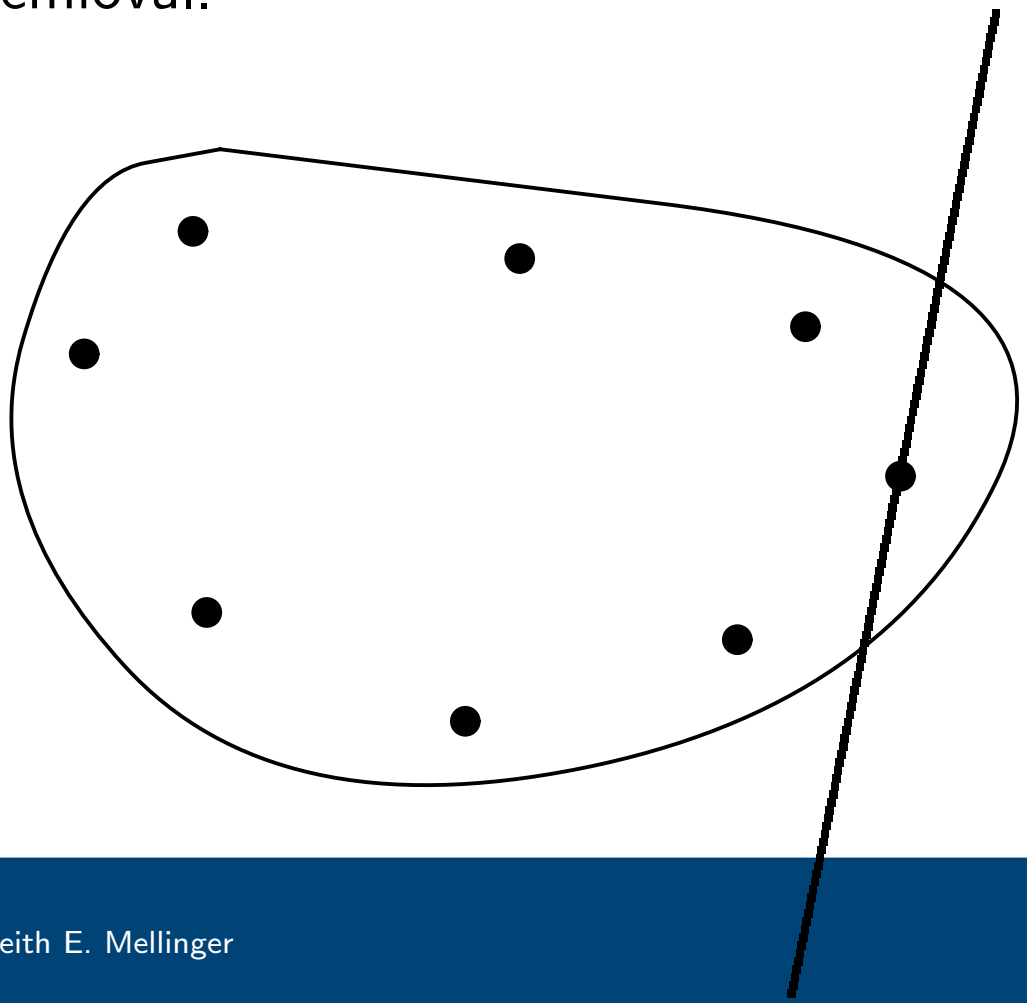

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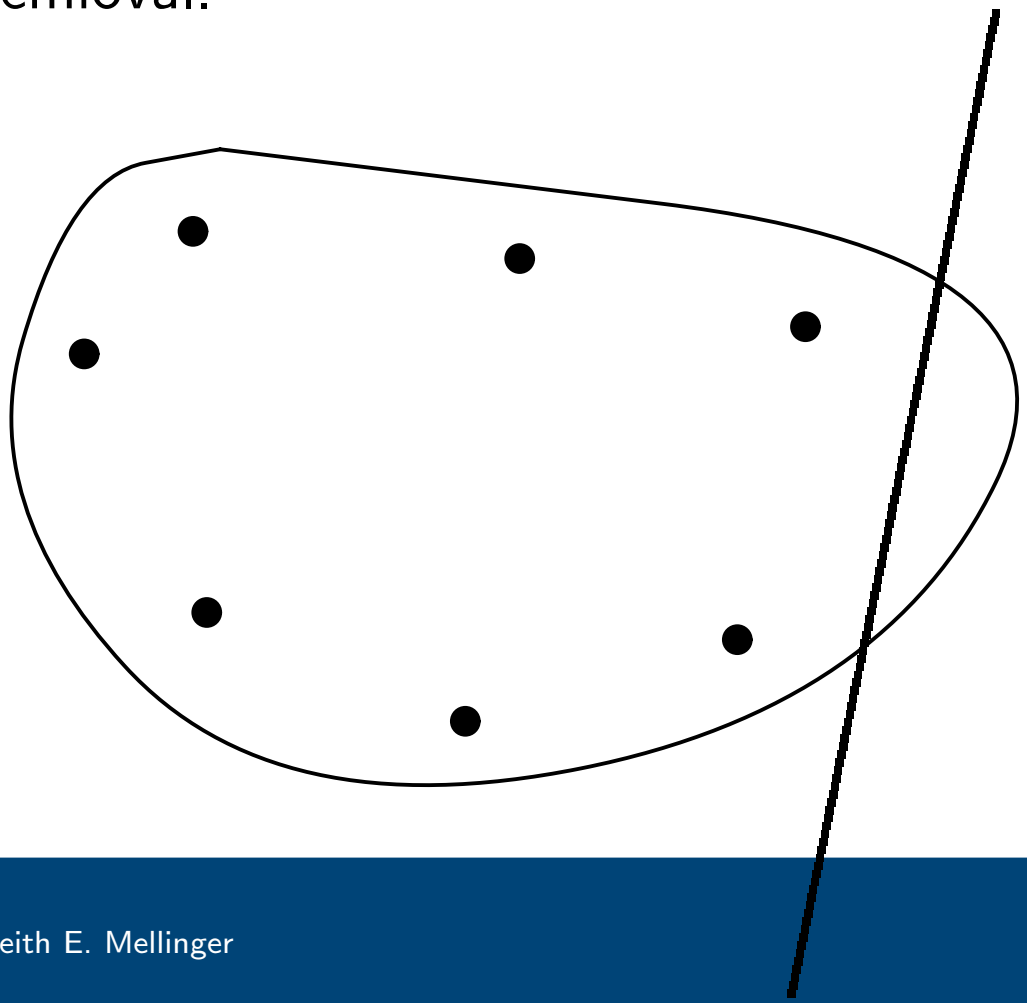

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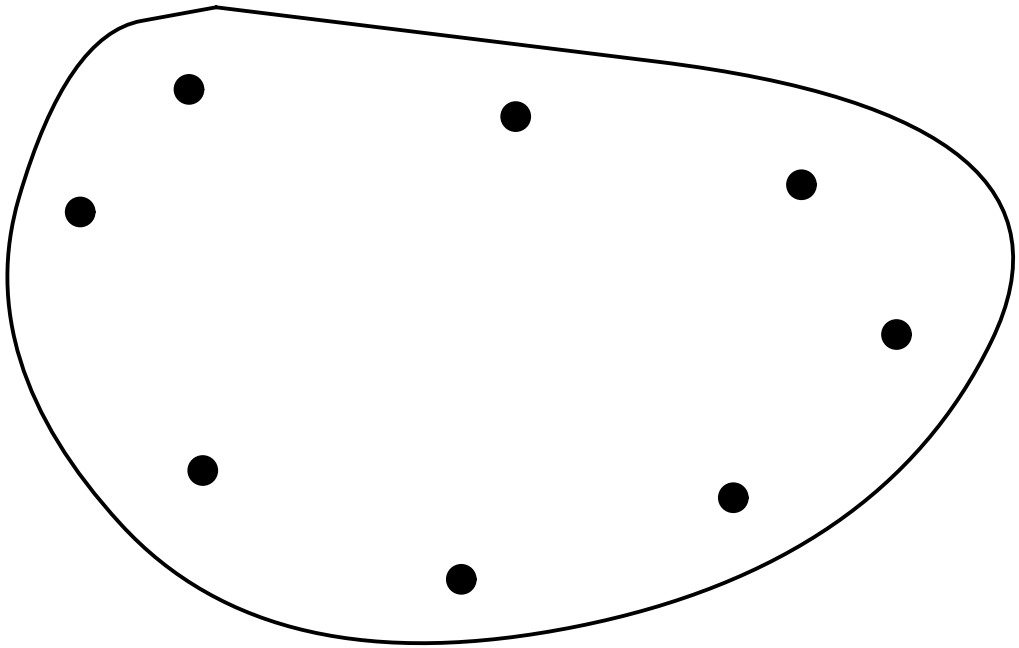

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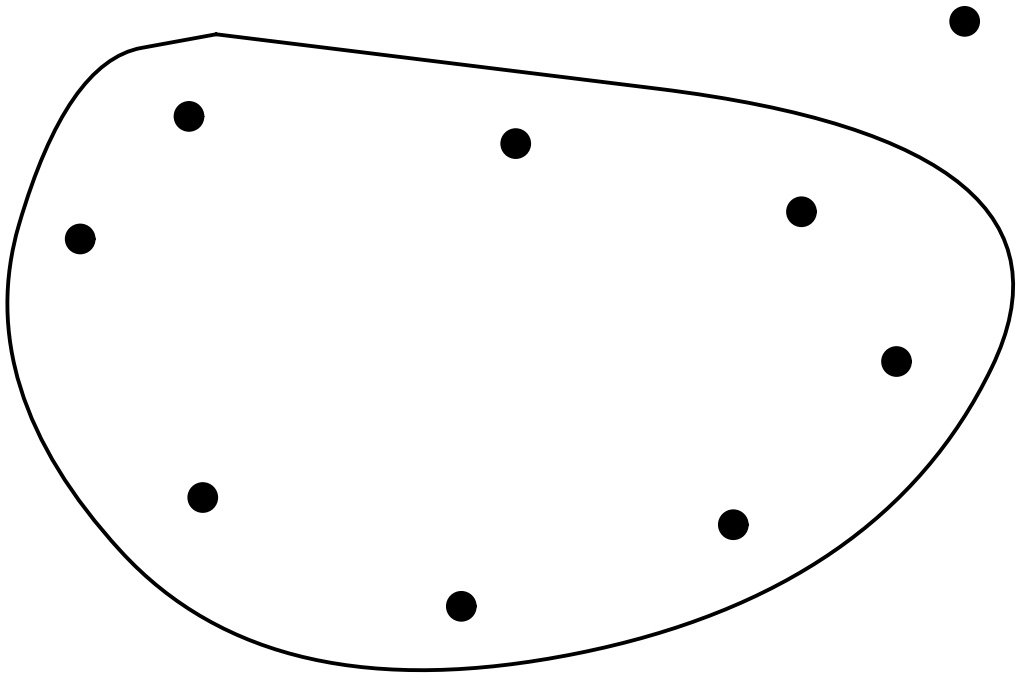

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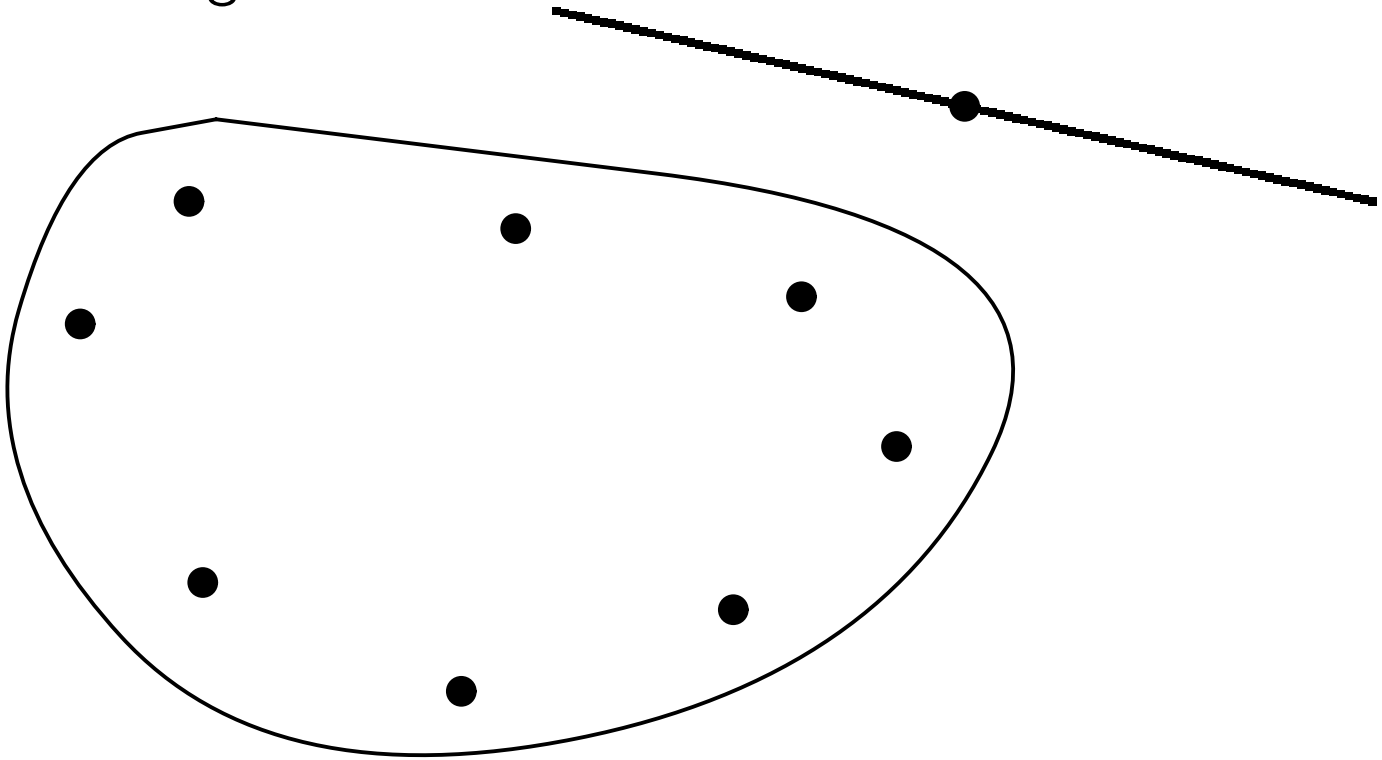

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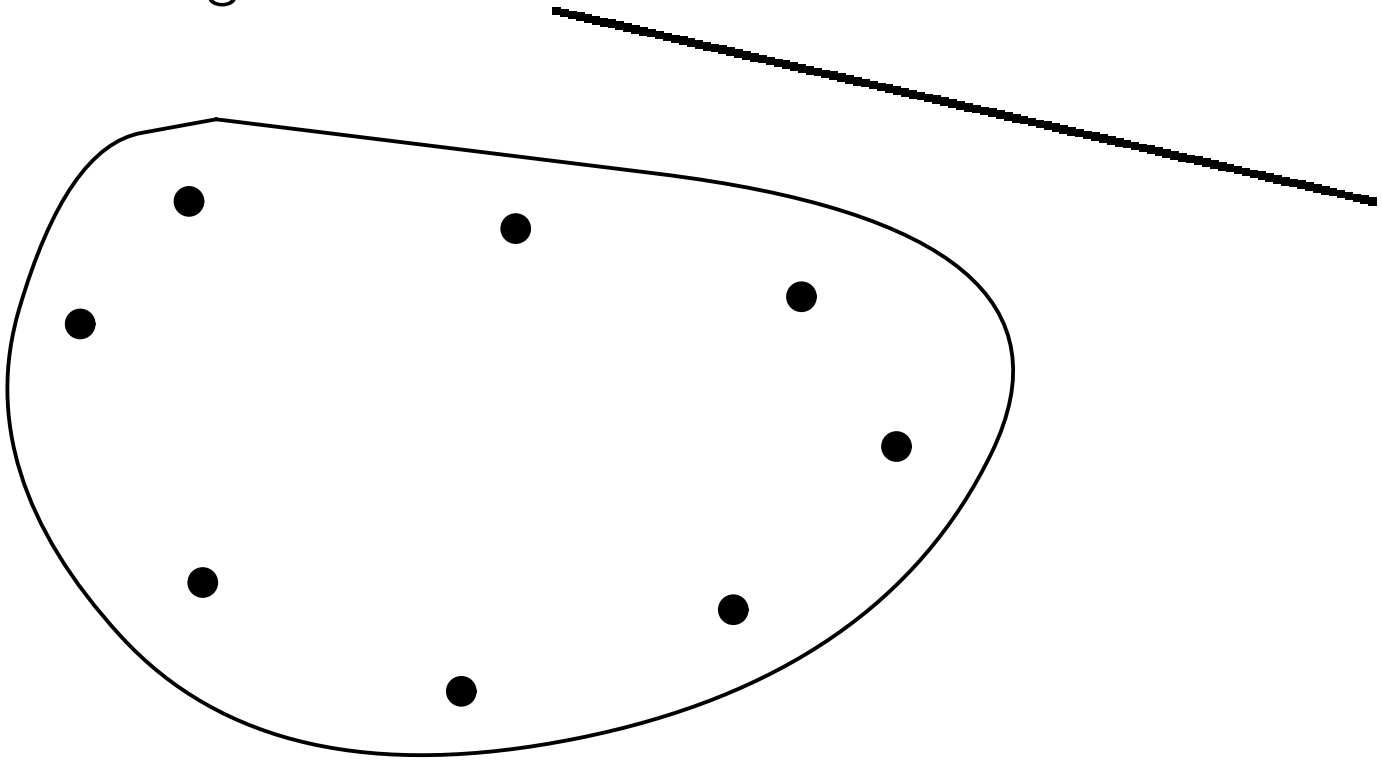
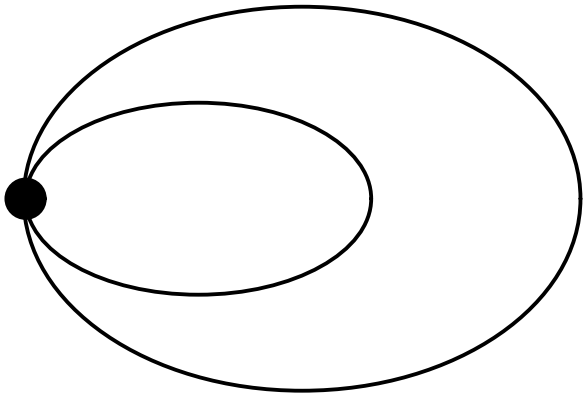


tangent pencil



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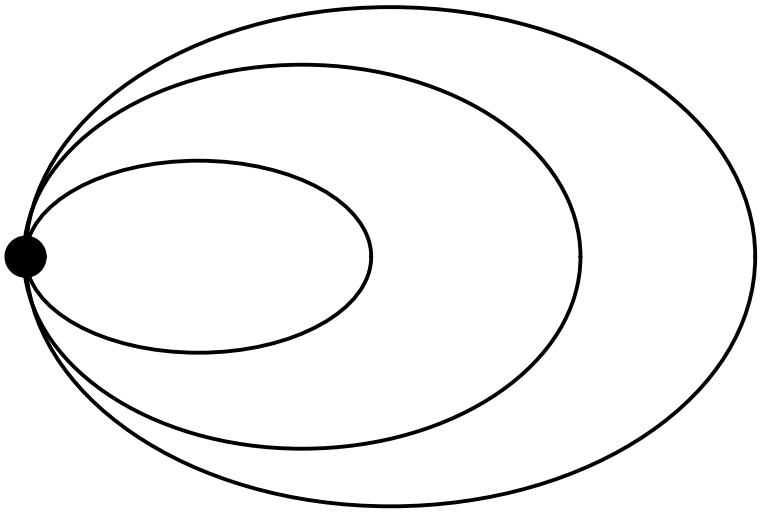

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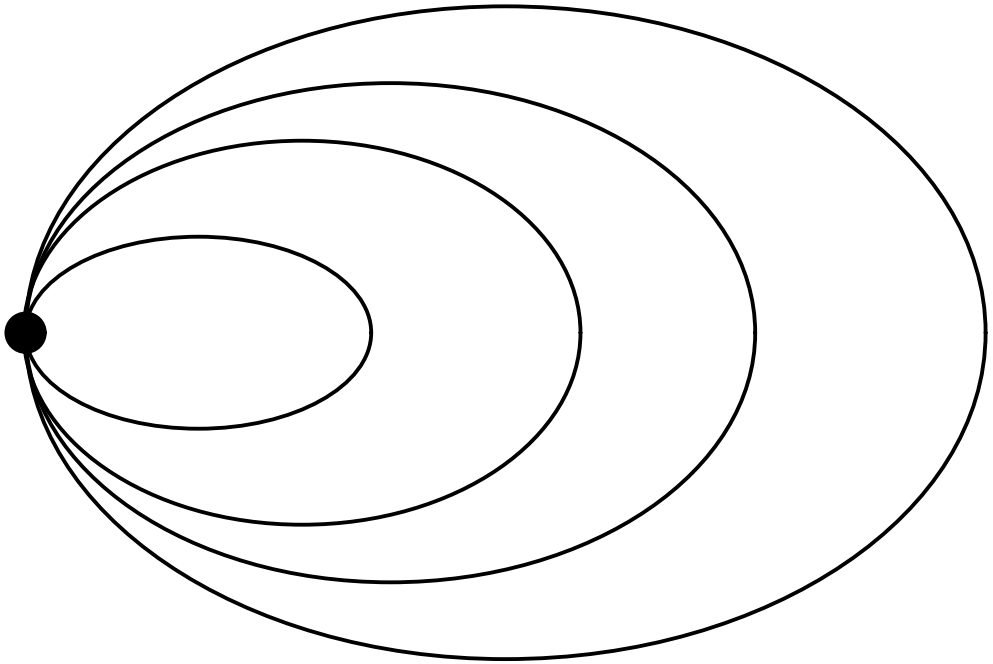

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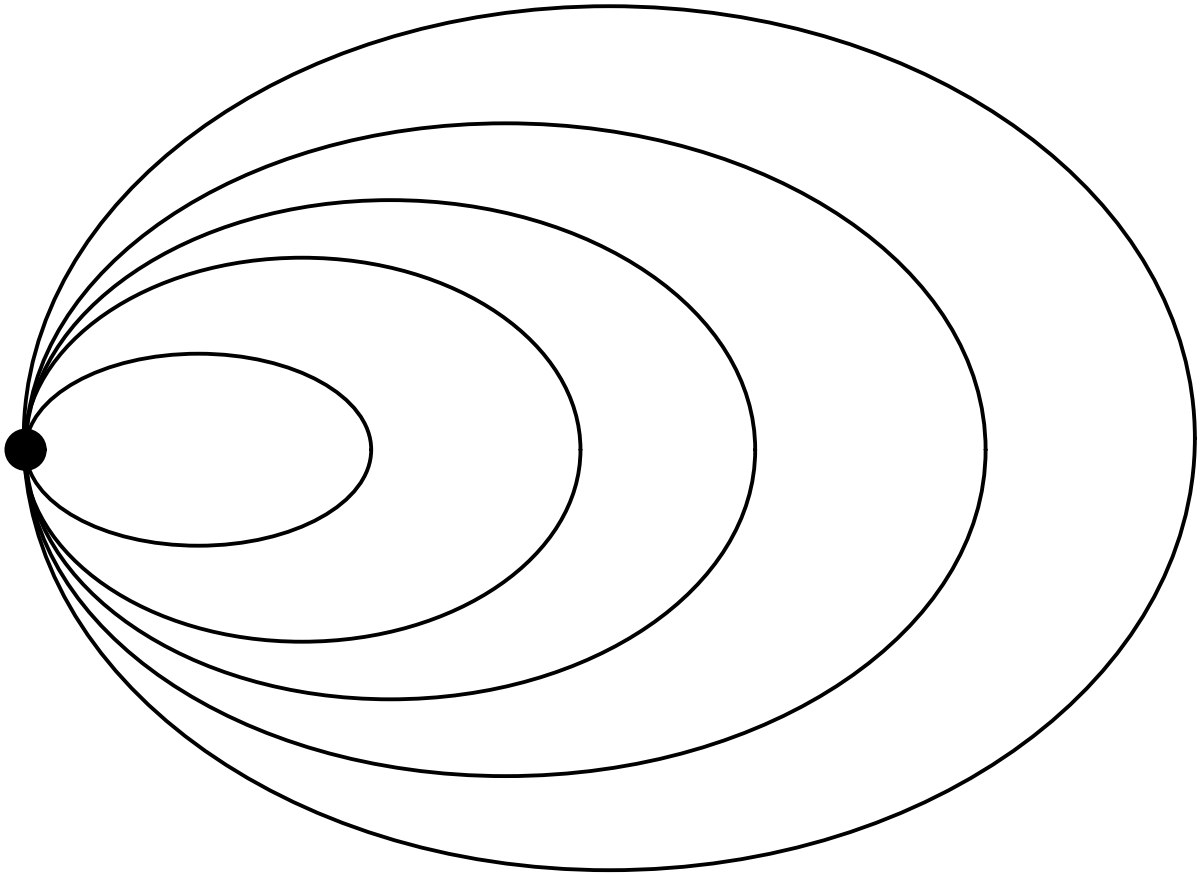

algebraic description

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In the arguments, Szőnyi shows that all points are "essential." In other words, no point can be removed without destroying the blocking property. This implies that there are unique tangent lines at each of the points.

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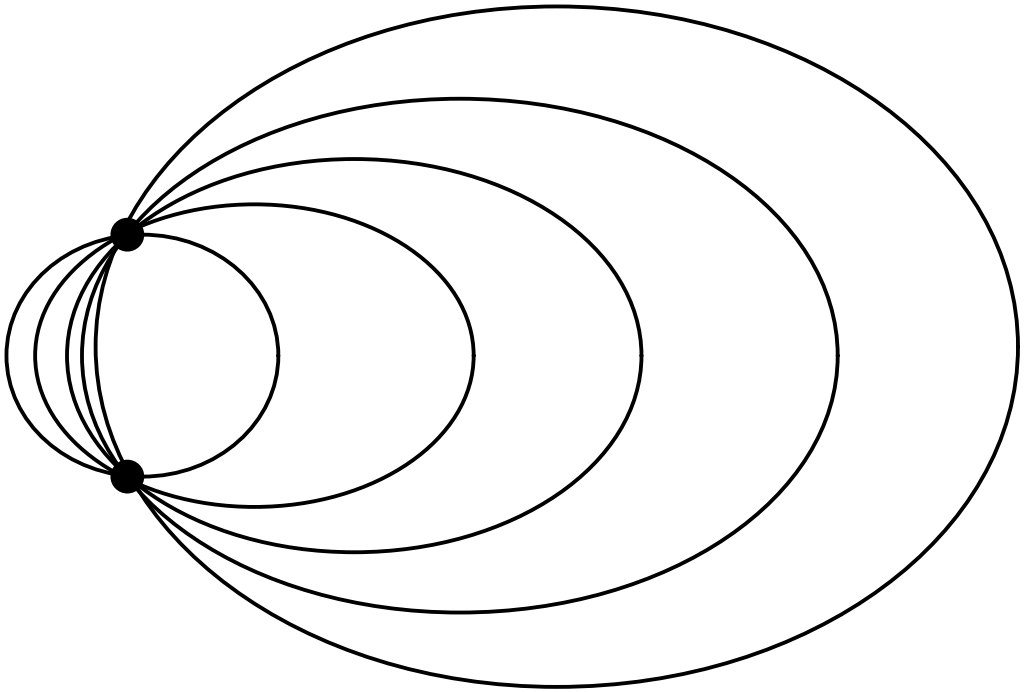
Was there any reason to start with a pair of tangent conics?

other pencils



semiovals from unions of conics

Theorem: Let $\mathcal{B} = \bigcup_{i=1}^{k} \mathcal{C}_i$ be a semioval in $PG(2, q)$ that is the union of nondegenerate conics $\mathcal{C}_i$. Then $\mathcal{B}$ is isomorphic to one of the following sets:

- ✔ a conic (note that this is the only possibility when q is even), or
- ✔ a union of at most $\sqrt{q}$ conics all lying in a common pencil, or
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So the semiovals that can be written as a union of conics are completely classified.

BSOs containing conics

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Theorem: In $\pi = PG(2, q^2)$, q odd, suppose $\mathcal{U}$ is a unital and $\mathcal{C}$ is a conic with commuting polarities μ and σ, respectively. $\mathcal{S} = \mathcal{C} \cup (\mathrm{Int}(\mathcal{C}) \cap \mathcal{U})$ is a blocking semioval in π.

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Theorem: Suppose $q \equiv 1 \pmod 4$, and let $A \subset GF(q)^*$ be a maximal subset of nonzero squares whose pairwise differences are nonsquares. Moreover suppose there exists no $d \in \not\square$ such that $k - d \in \square$ for all $k \in A$. Then $\mathcal{S} = \bigcup_{k \in A} \mathcal{C}_k \cup \{(1, a, 0) : a \in \not\square\}$ is a blocking semioval.

Conclusions

- ✔ with Jeremy Dover and Kenneth Wantz, Blocking semiovals containing conics, *Adv. Geom.* **13**: 1 (January 2013) 29-40.
- ✔ with Jeremy Dover, Semiovals from unions of conics, *Innov. Incidence Geom.*, **12** (2011) 61-83.

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Thanks!

Conclusions

- ✔ with Jeremy Dover and Kenneth Wantz, Blocking semiovals containing conics, *Adv. Geom.* **13**: 1 (January 2013) 29-40.
- ✔ with Jeremy Dover, Semiovals from unions of conics, *Innov. Incidence Geom.*, **12** (2011) 61-83.

Thanks!

Questions?