Minimal Kakeya Sets

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(joint work with Jeremy Dover and Kelly Scott)

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What is a blocking set?

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A blocking set $\mathcal{S}$ in a projective plane π is a set of points such that every line in π contains at least one point in $\mathcal{S}$ and one point not in $\mathcal{S}$. Therefore $\mathcal{S}$ intersects every line in π but $\mathcal{S}$ contains no lines.

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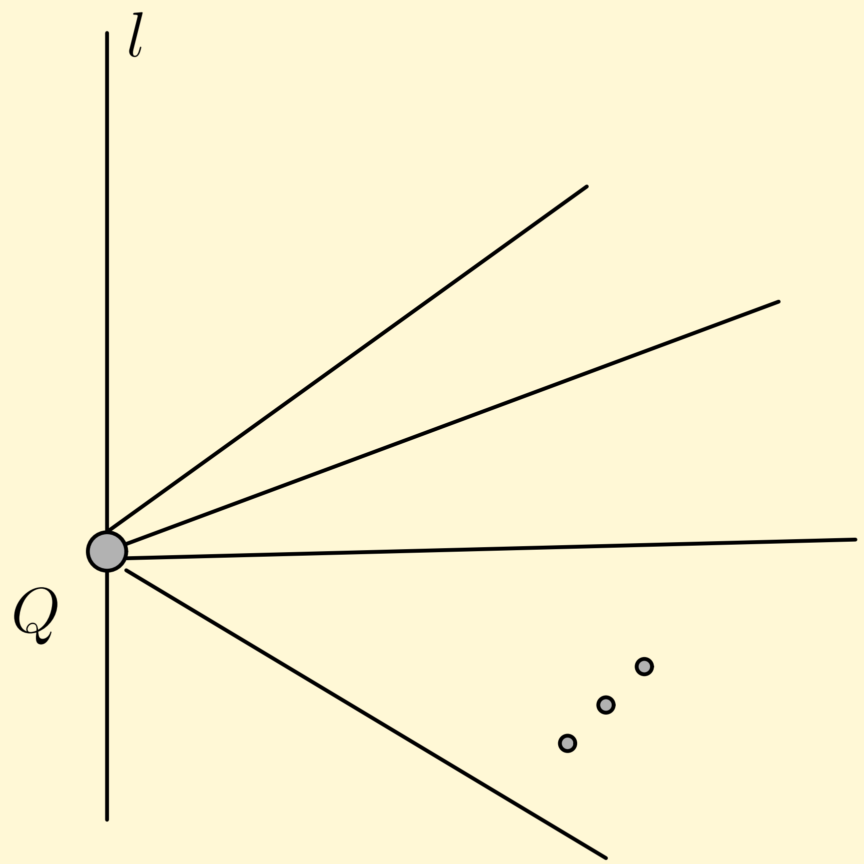

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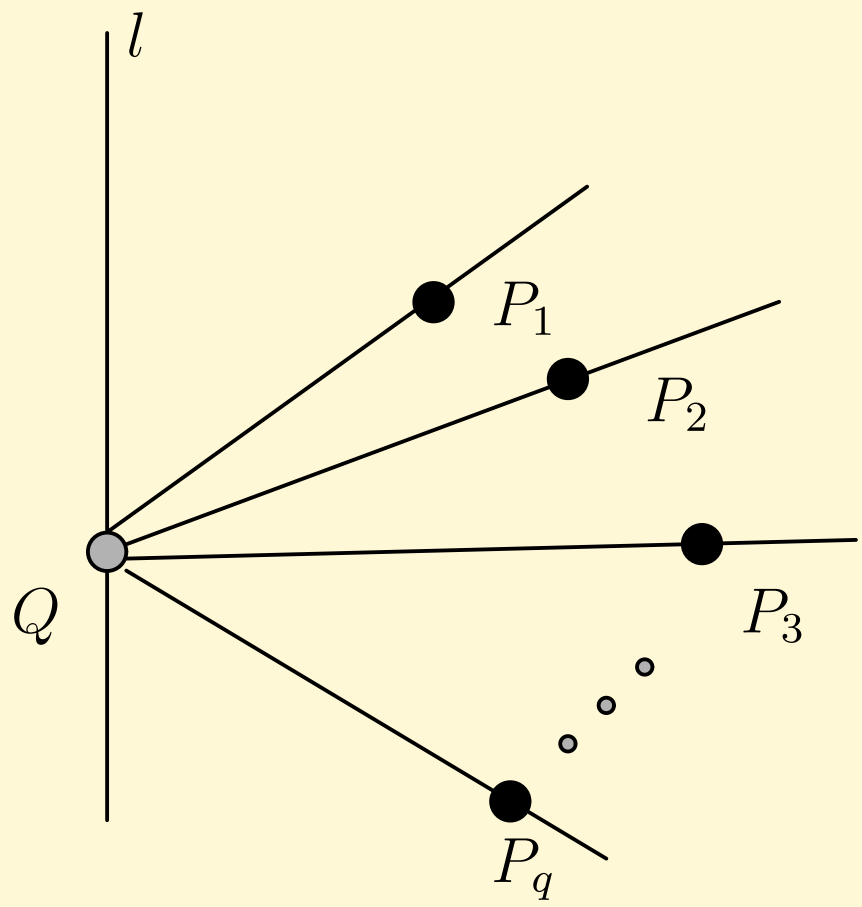
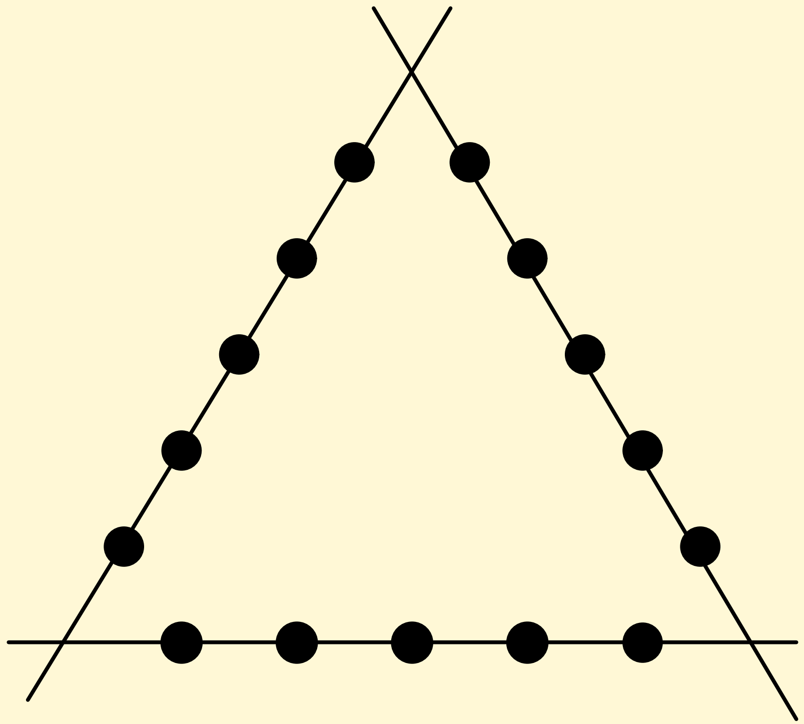


Example #2: Vertexless Triangle



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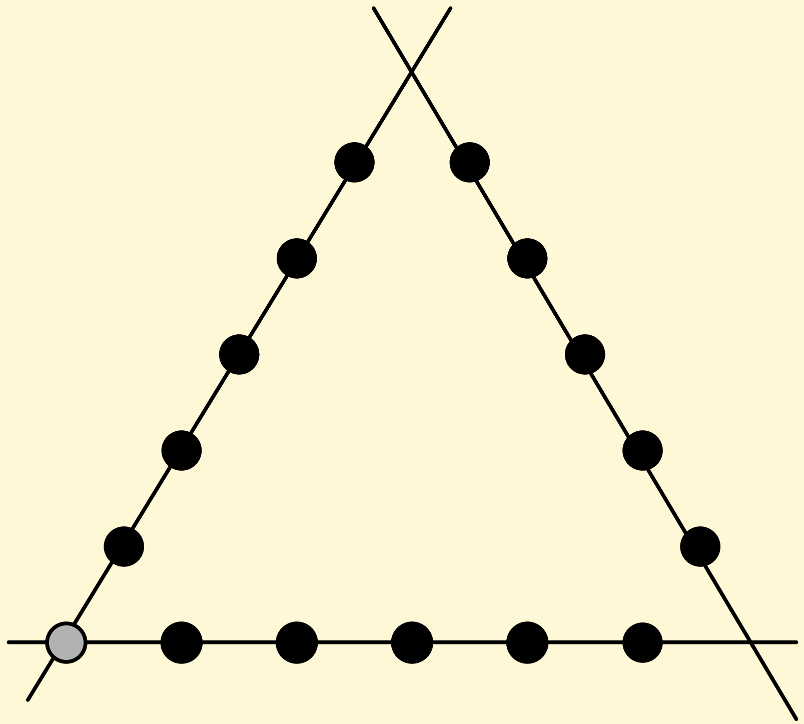

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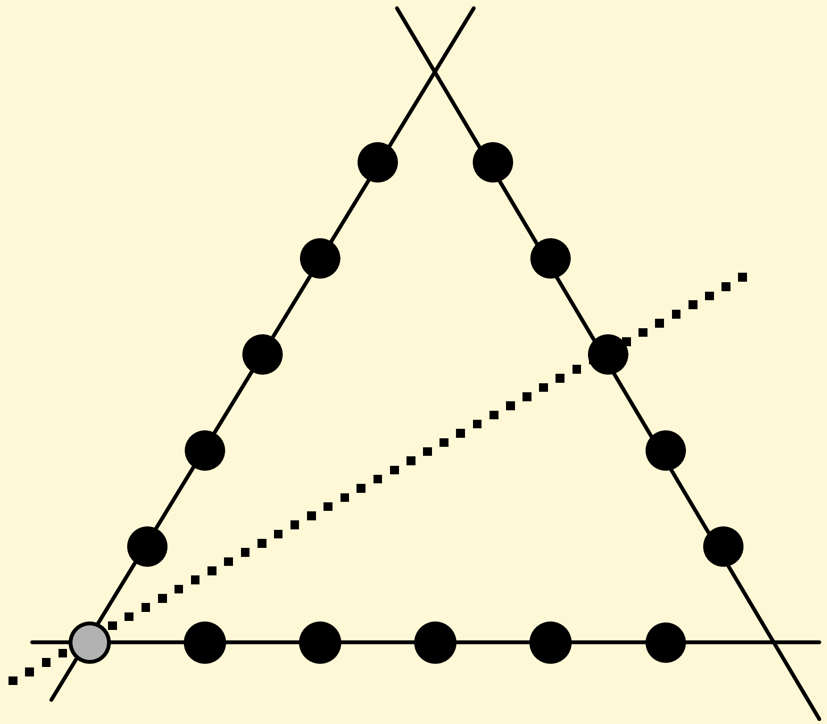

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A natural definition

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Definition: An anti-blocking set $\mathcal{A}$ is a set of points such that:

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Definition: An anti-blocking set $\mathcal{A}$ is a set of points such that:

- $\mathcal{A}$ does not block all lines, and

A natural definition

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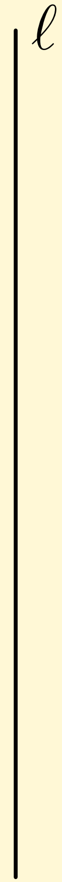
Definition: An anti-blocking set $\mathcal{A}$ is a set of points such that:

- $\mathcal{A}$ does not block all lines, and
- $\mathcal{A}$ is not a subset of any blocking set.

Example #1: Complement of a Line



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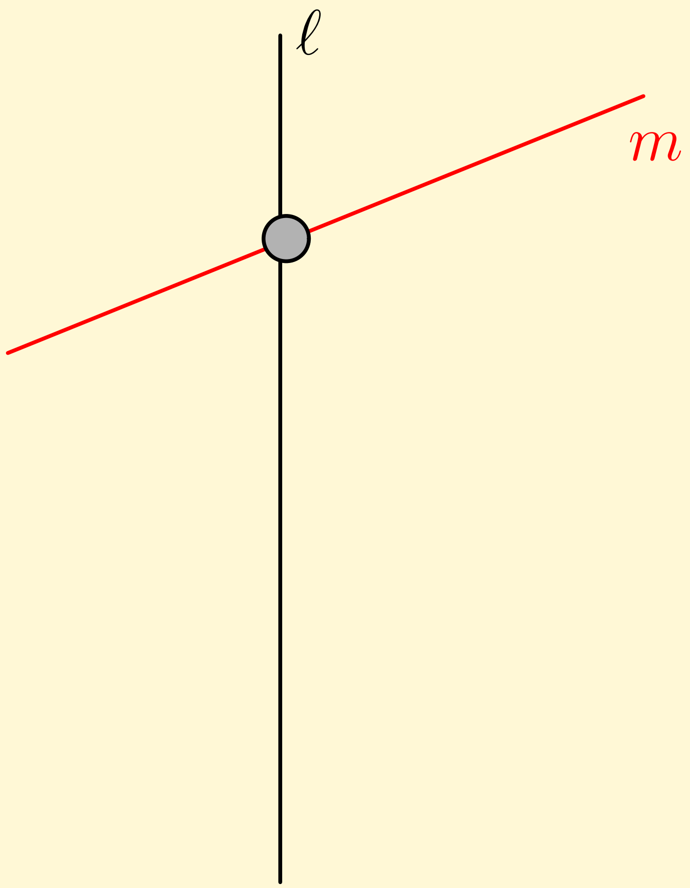

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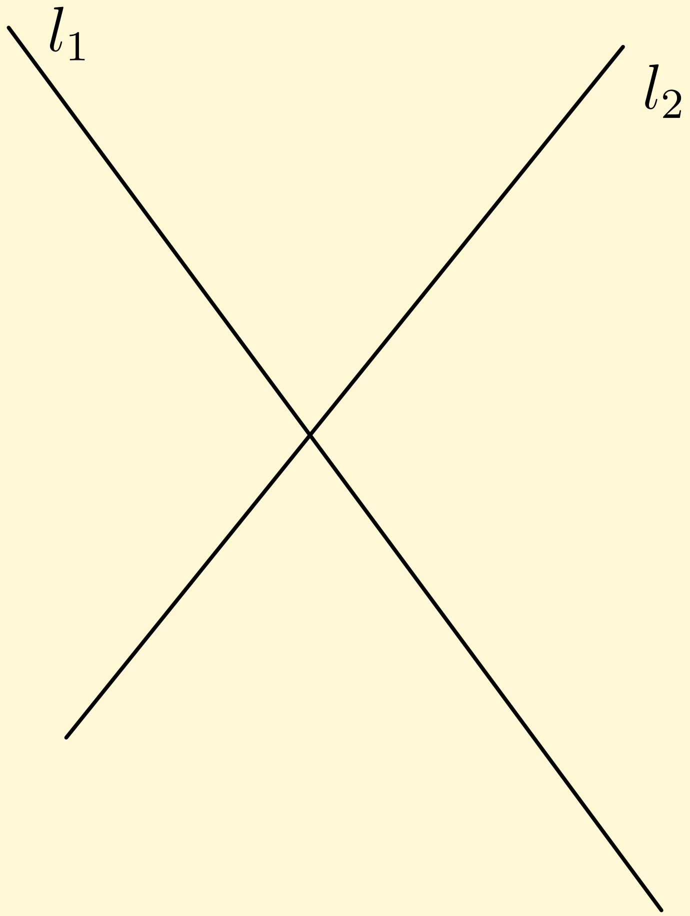
Example #2: Complement of 2 Lines



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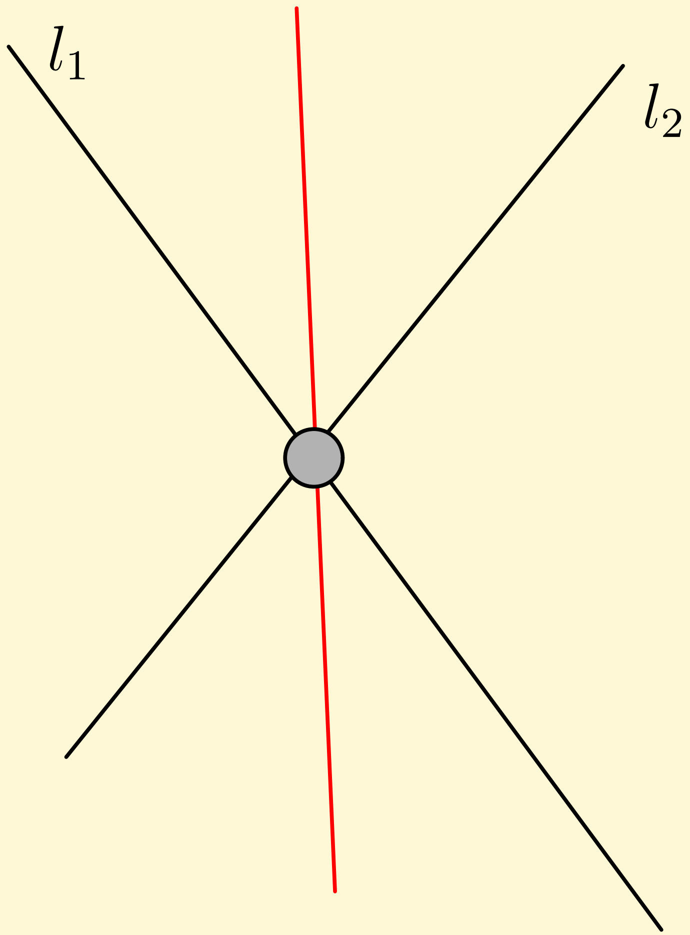

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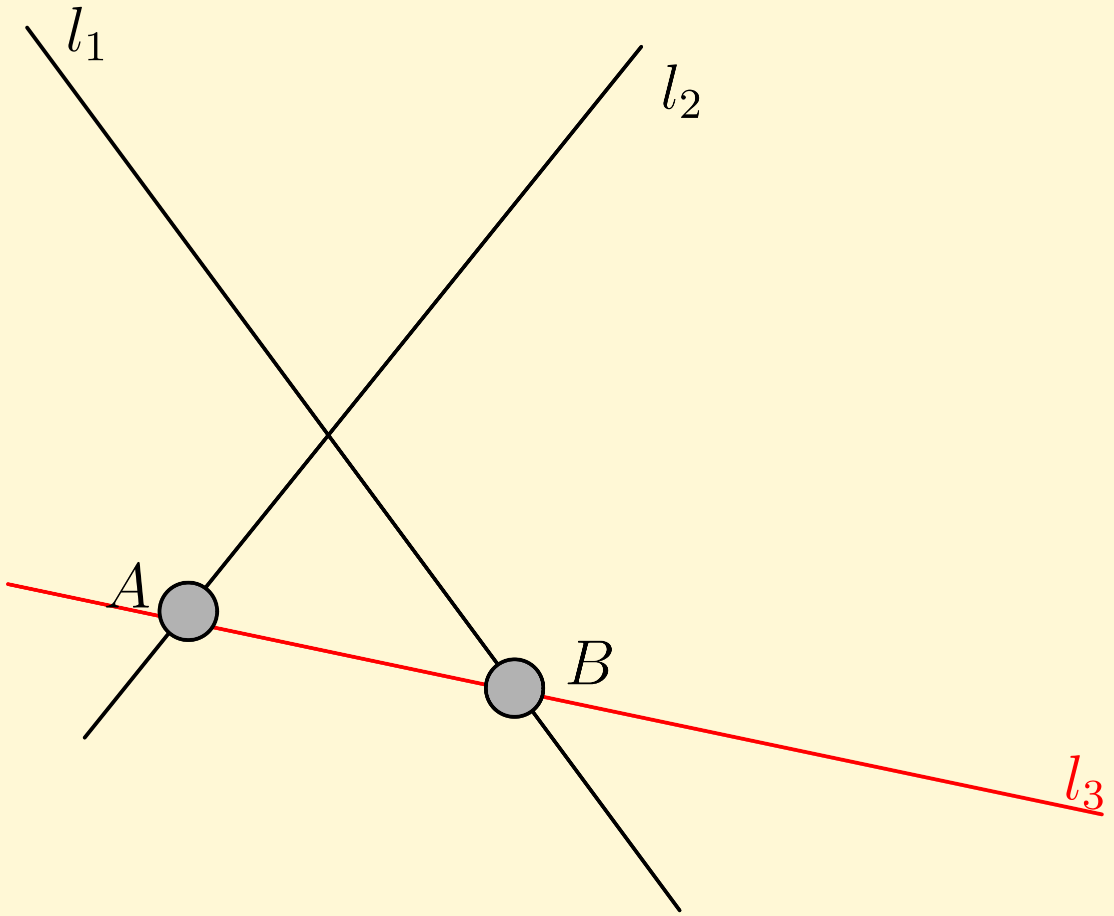

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Theorem Let $\mathcal{A}$ be an anti-blocking set and $\mathcal{L}$ the set of all lines that do not contain a point of $\mathcal{A}$. Then $|\mathcal{L}| \leq 2$.

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Theorem Let $\mathcal{A}$ be an anti-blocking set and $\mathcal{L}$ the set of all lines that do not contain a point of $\mathcal{A}$. Then $|\mathcal{L}| \leq 2$.

By way of contradiction, suppose $|\mathcal{L}| \not\leq 2$ so that $|\mathcal{L}| \geq 3$.

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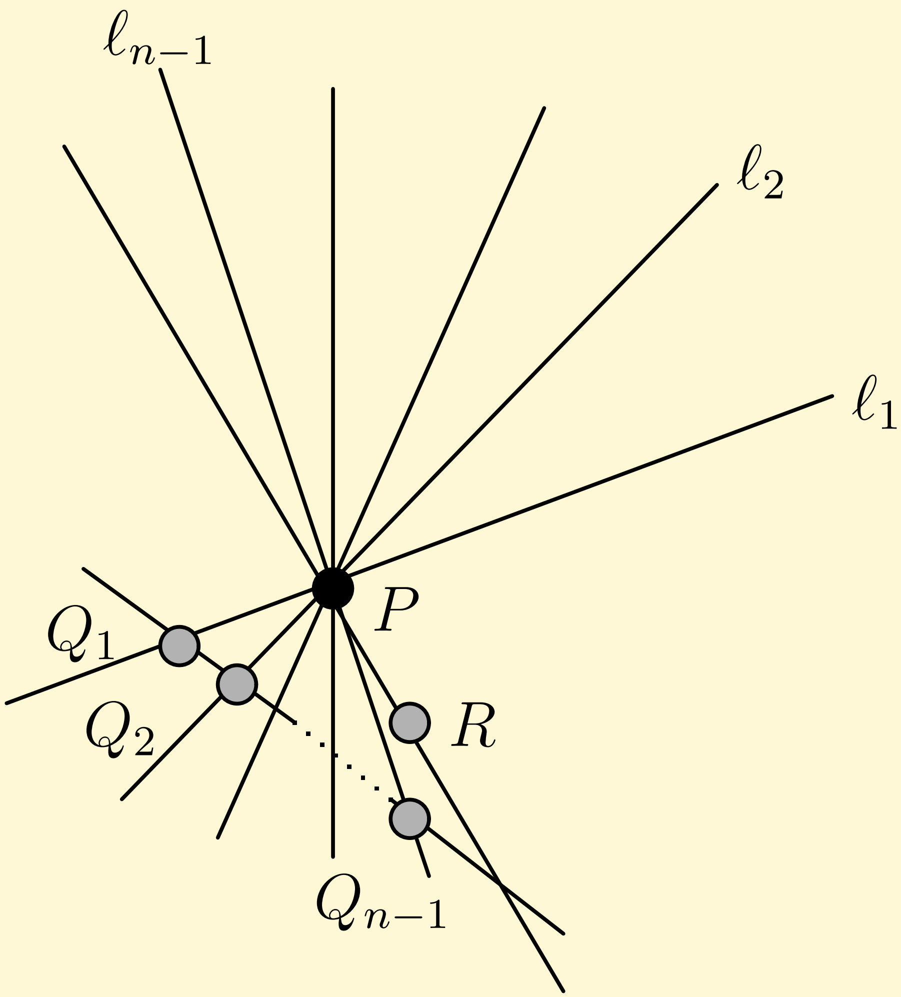
Theorem Let $\mathcal{A}$ be an anti-blocking set and $\mathcal{L}$ the set of all lines that do not contain a point of $\mathcal{A}$. Then $|\mathcal{L}| \leq 2$.

By way of contradiction, suppose $|\mathcal{L}| \not\leq 2$ so that $|\mathcal{L}| \geq 3$.

We show that points can be added to $\mathcal{A}$ in such a way as to build a blocking set. Therefore, $\mathcal{A}$ is in fact a subset of a blocking set and so is not an anti-blocking set. This will prove our result.



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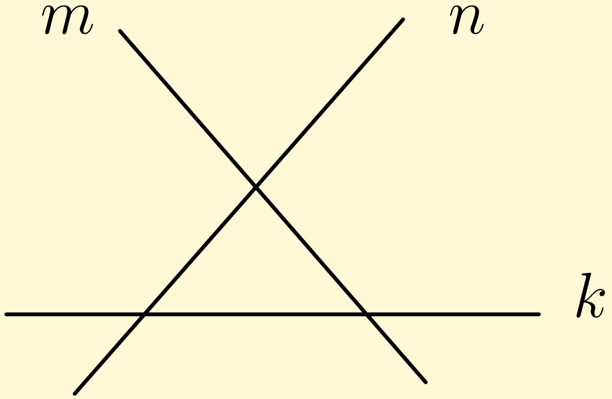

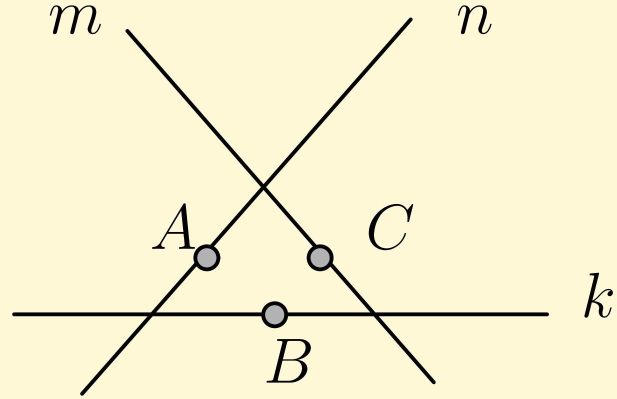

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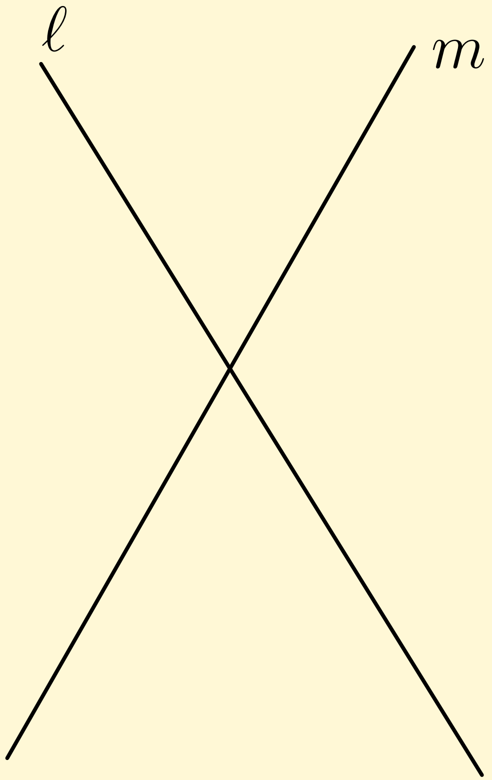
Theorem Let $\mathcal{A}$ be an anti-blocking set. If $\mathcal{A}$ is a subset of the complement of two lines, then $\mathcal{A}$ is exactly equal to the complement of two lines.



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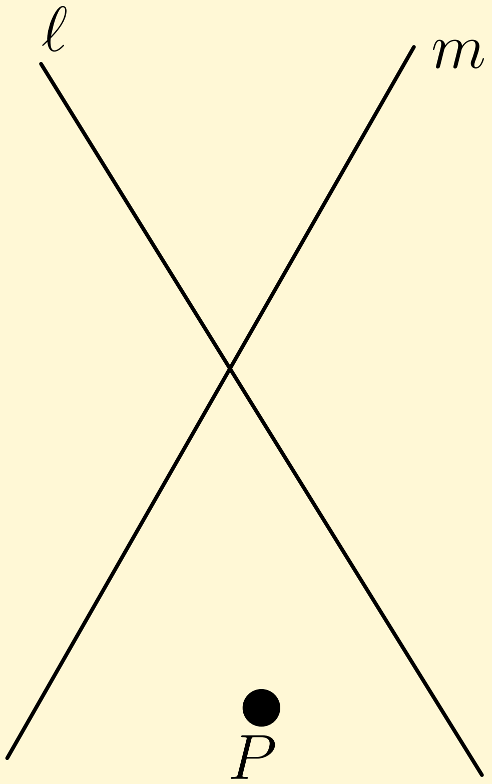

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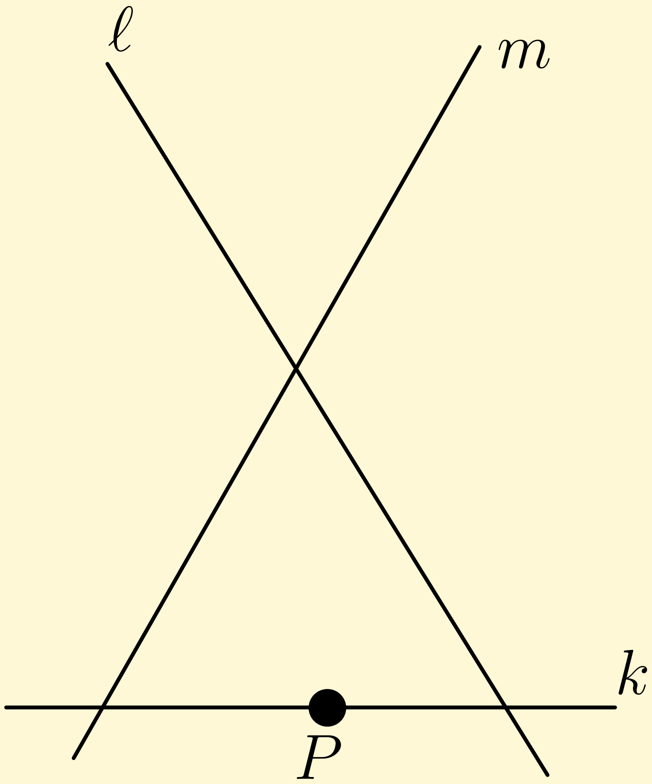

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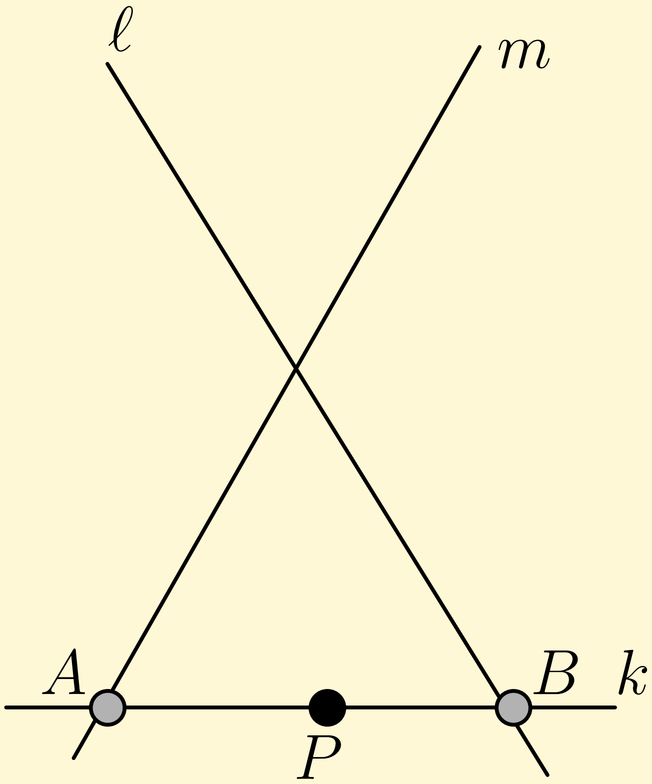

A characterization

One of our original ideas was to use pencils of conics to try to construct anti-blocking sets.

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A characterization

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Our idea relies on a characterization of anti-blocking sets:

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A characterization

One of our original ideas was to use pencils of conics to try to construct anti-blocking sets.

Our idea relies on a characterization of anti-blocking sets:

Theorem: Let ℓ_∞ be a line of the projective plane π. Suppose $\mathcal{A} \subseteq (\pi \setminus \ell_\infty)$, and $\mathcal{A}$ is not the complement of two lines. Then, $\mathcal{A}$ is an anti-blocking set if and only if for every point $P \in \ell_\infty$, there is a line ℓ ($\neq \ell_\infty$) through P such that $(\ell \setminus \{P\}) \subset \mathcal{A}$.

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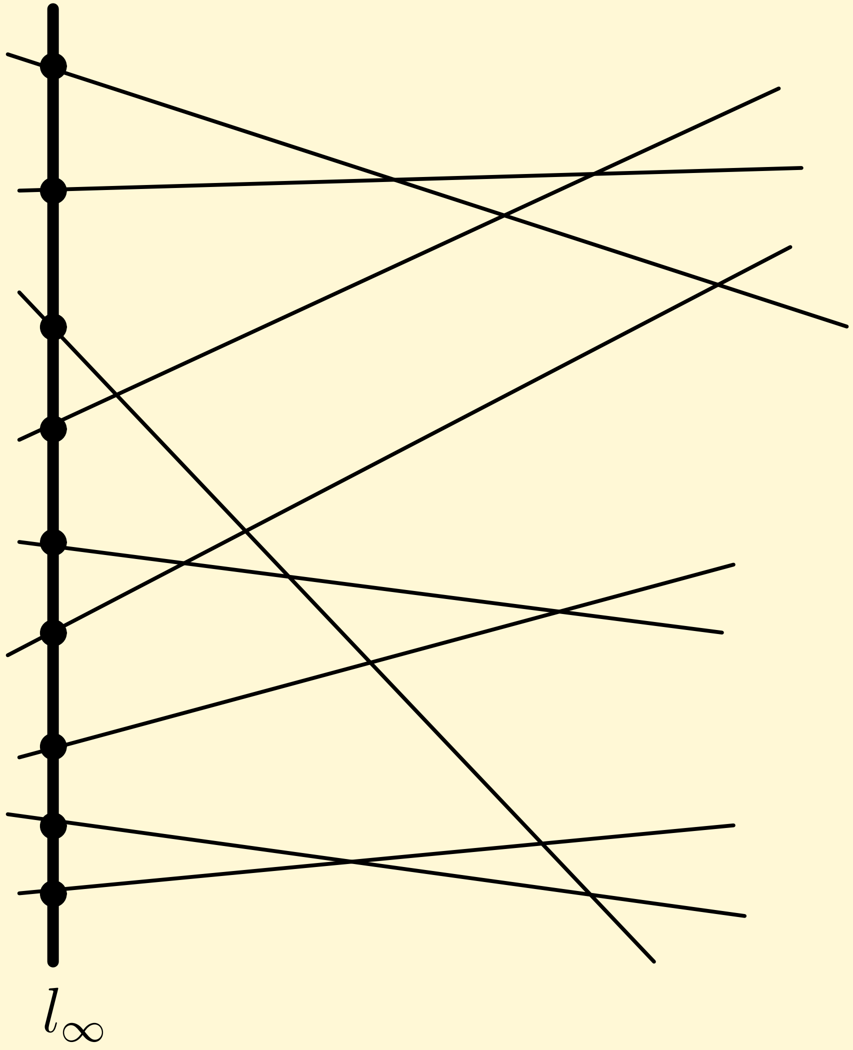

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The finite field Kakeya problem, proposed by Thomas Wolff, asks for the smallest subset of points of $AG(n, q)$ which contains a line in every direction.

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$$\mathcal{E} = (Ext \cup \ell \cup \mathcal{C}) \setminus \ell_\infty$$

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$$\mathcal{E} = (\bigcup t_i \cup \ell) \setminus \ell_\infty$$

A Besikovitch Set

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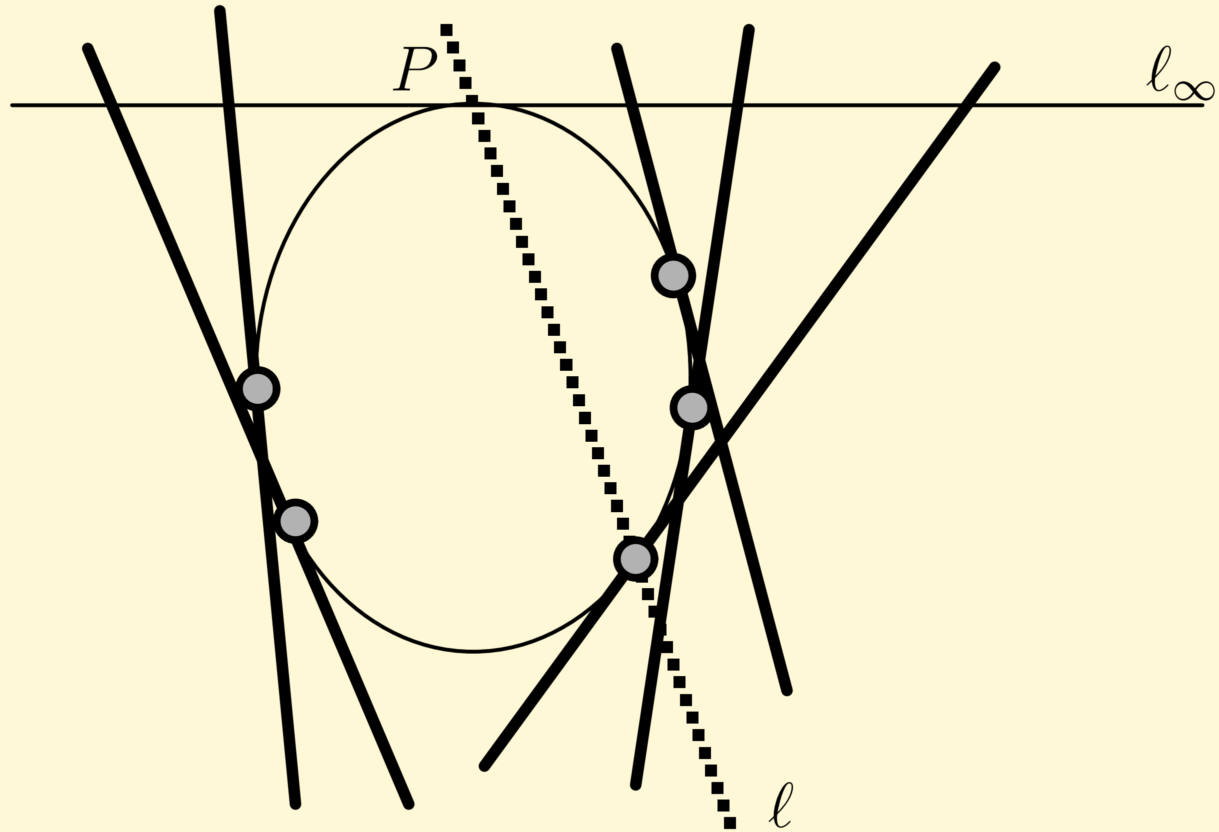

the minimal problem

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With the minimum problem solved, one naturally turns to the *minimal* problem.

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A bound on the largest minimal Kakeya set is straight-forward to calculate. Basically, you take a line through every point of a fixed line ℓ and try to force them to overlap as little as possible.

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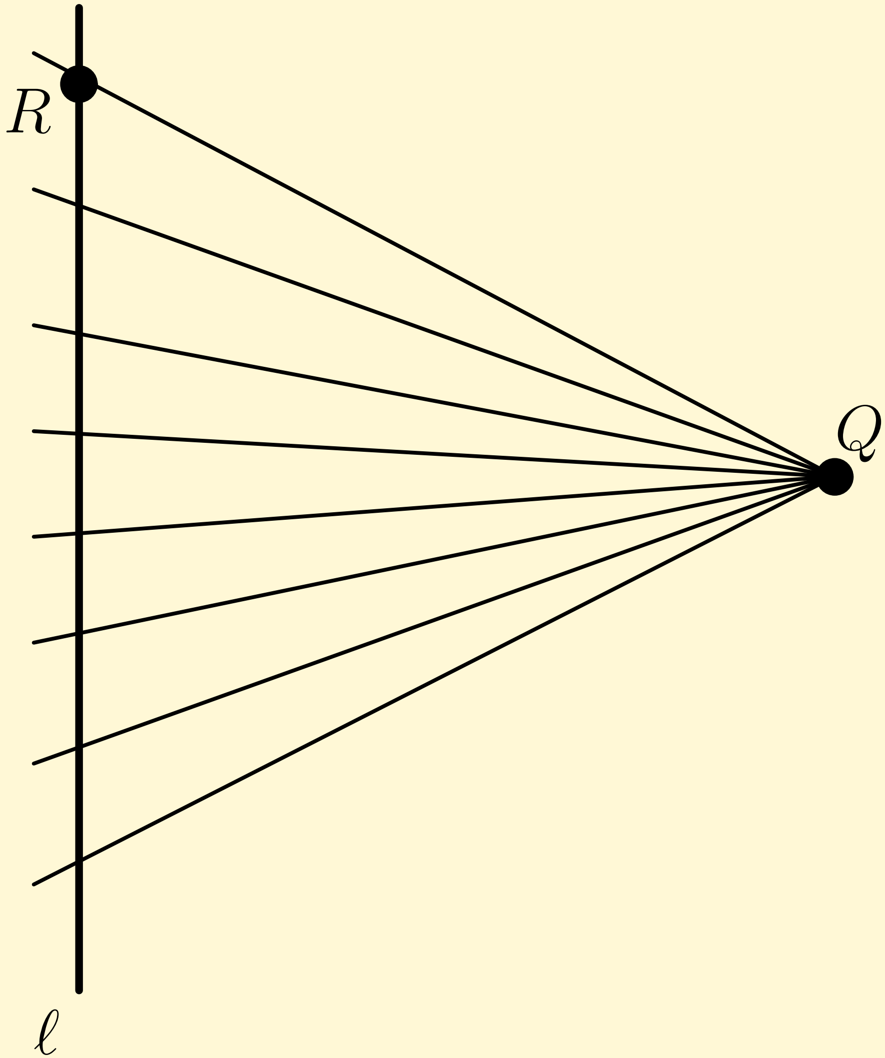
A bound on the largest minimal Kakeya set is straight-forward to calculate. Basically, you take a line through every point of a fixed line ℓ and try to force them to overlap as little as possible.

A simple count shows

$$|\mathcal{S}| \leq q^2 - q + 1.$$

construction of the maximum MKS



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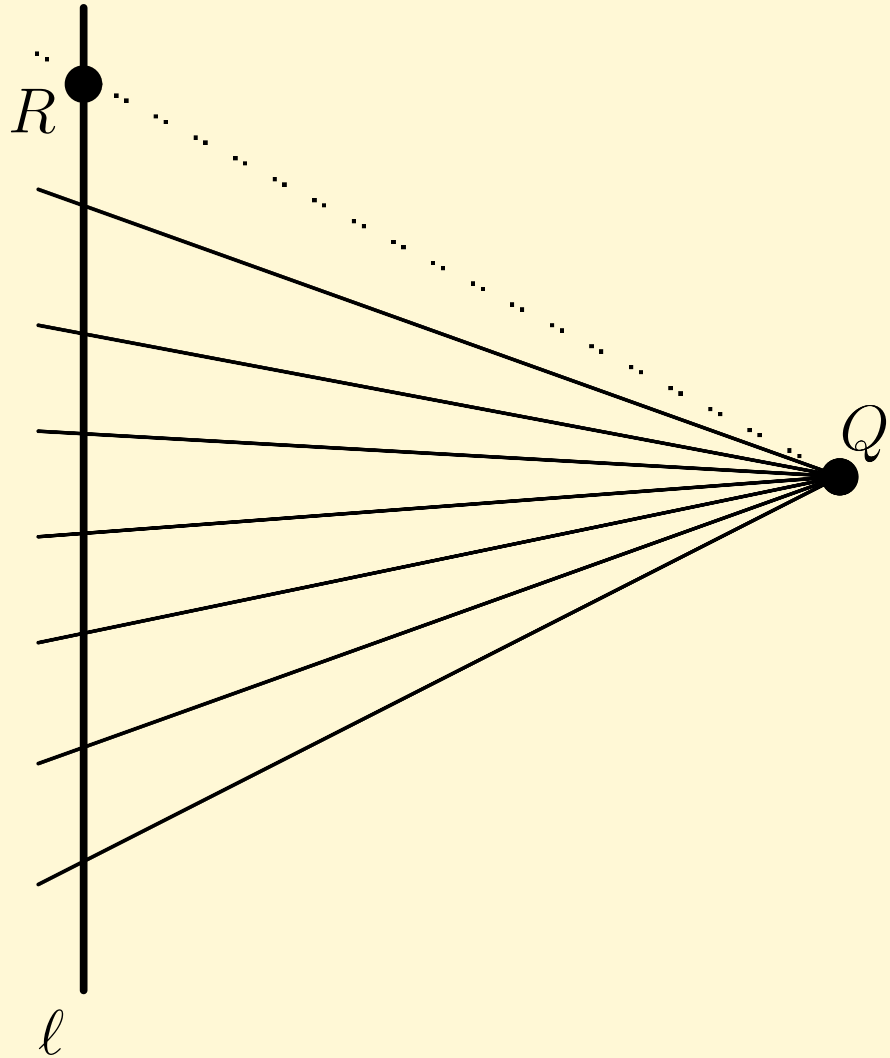

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$PG(2,3)$**:** The only minimal Kakeya set has size 7.

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$PG(2,3)$: The only minimal Kakeya set has size 7.

$PG(2,4)$: There exist minimal Kakeya sets of size 10 (the points on $q+1$ lines of a dual hyperoval, less those points on the excluded line of the dual hyperoval) and size 13.

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$PG(2,5)$: There exist minimal Kakeya sets of size 17 (the Blokhuis-Mazzocca example of minimum size in $PG(2,5)$), size 21 (the maximum example), and...

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$PG(2,3)$: The only minimal Kakeya set has size 7.

$PG(2,4)$: There exist minimal Kakeya sets of size 10 (the points on $q+1$ lines of a dual hyperoval, less those points on the excluded line of the dual hyperoval) and size 13.

$PG(2,5)$: There exist minimal Kakeya sets of size 17 (the Blokhuis-Mazzocca example of minimum size in $PG(2,5)$), size 21 (the maximum example), and... one of size 18.

$\mathcal{P} = \{C_\lambda : \lambda \in GF(q) \cup \{\infty\}\}$, where

$$C_\lambda = V(\lambda x^2 + y^2 - sz^2)$$

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the algebraic pencil

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$\mathcal{P} = \{C_\lambda : \lambda \in GF(q) \cup \{\infty\}\}$, where

$$C_\lambda = V(\lambda x^2 + y^2 - sz^2)$$

By examining the computer generated example, we discovered that the minimal example of size 18 in $PG(2,5)$ was again a union of lines (less ℓ_∞), and that all of these lines were tangent lines to one of two conics in the algebraic pencil above.

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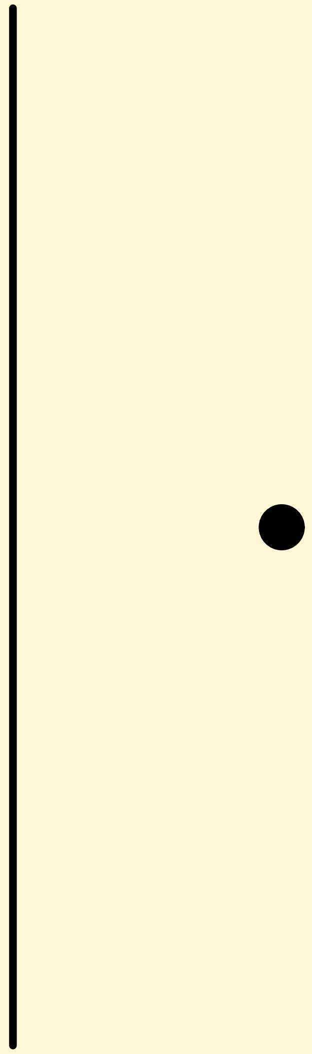

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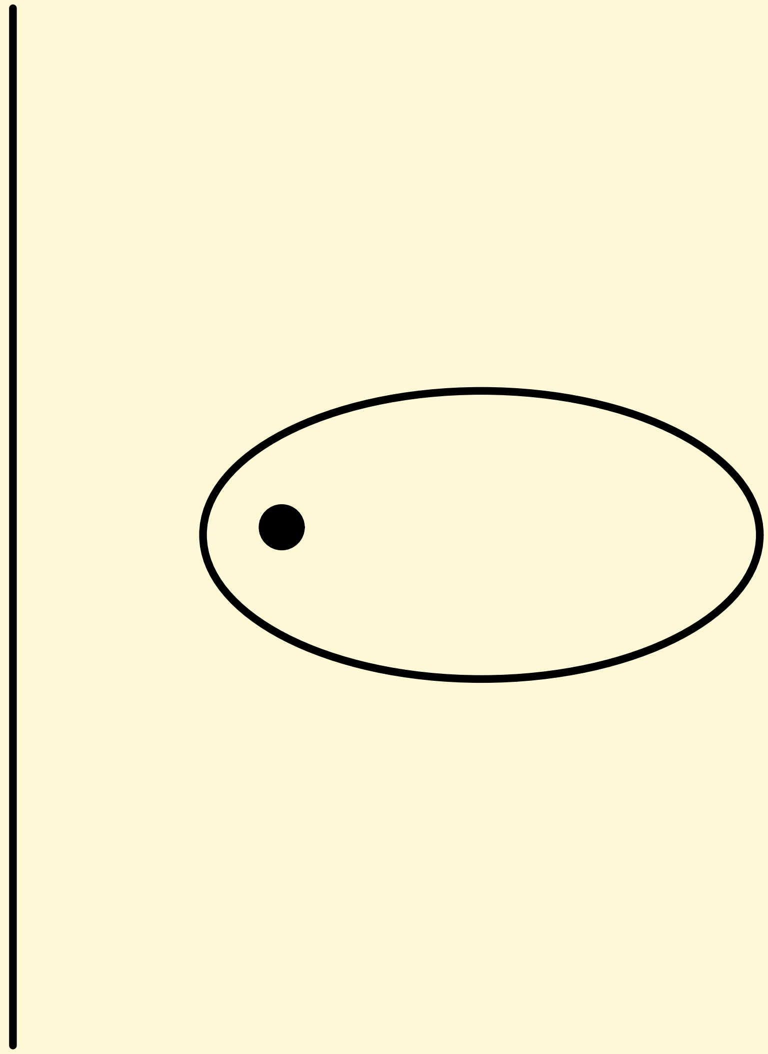

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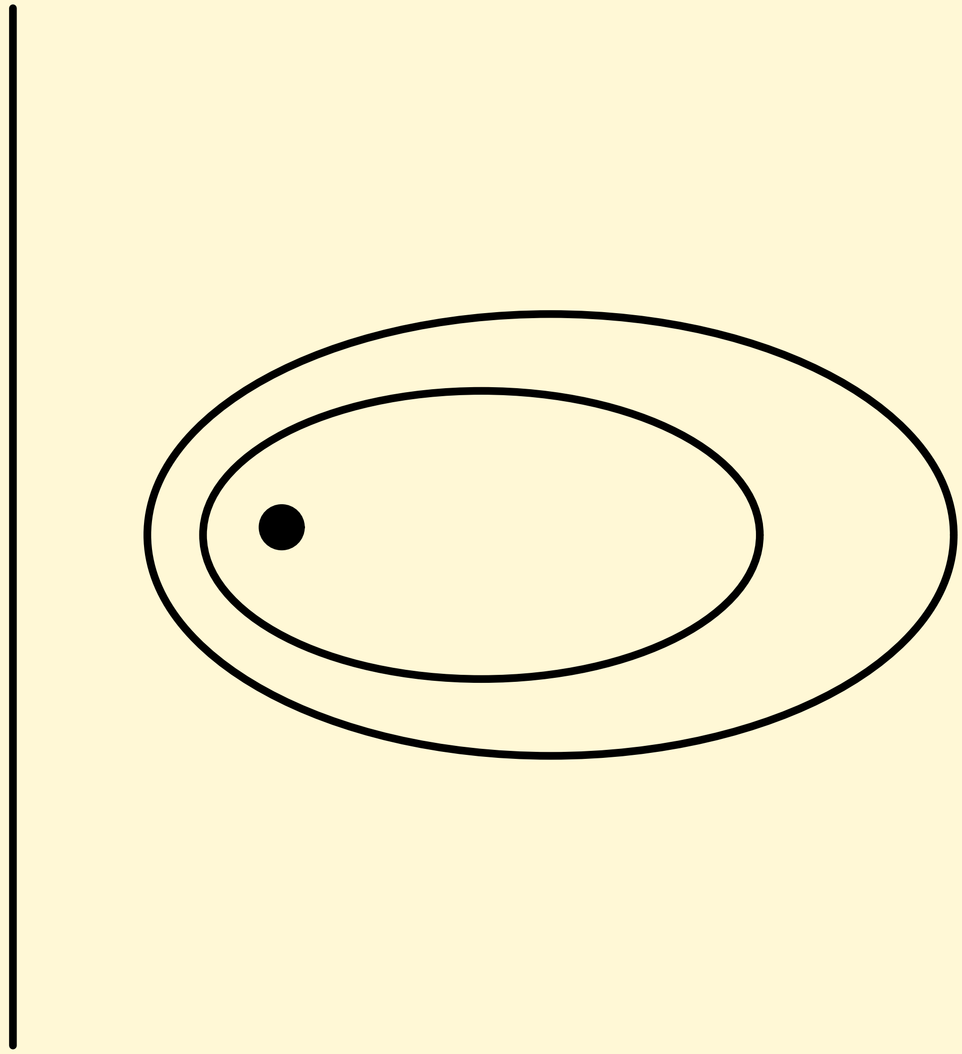

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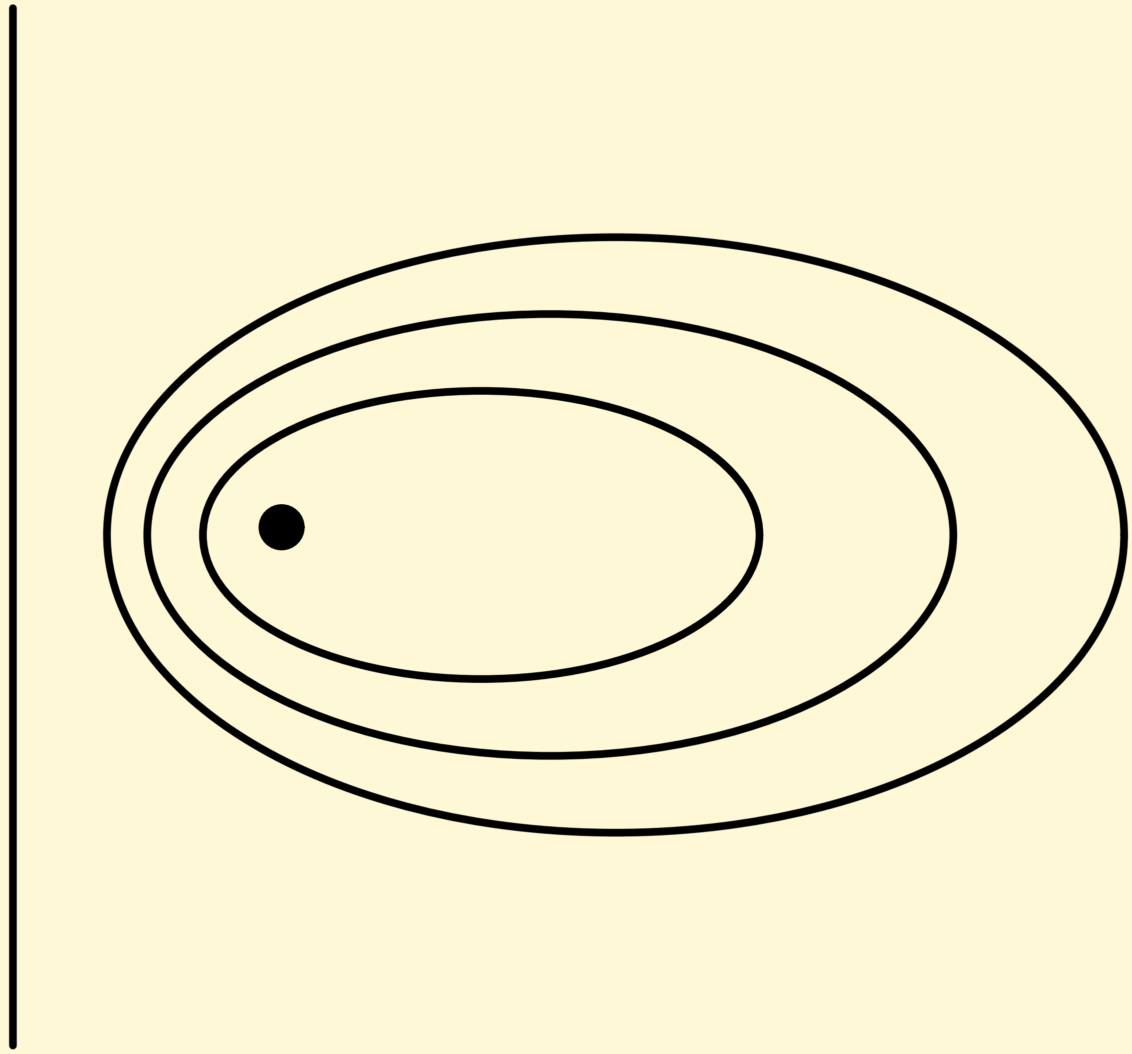

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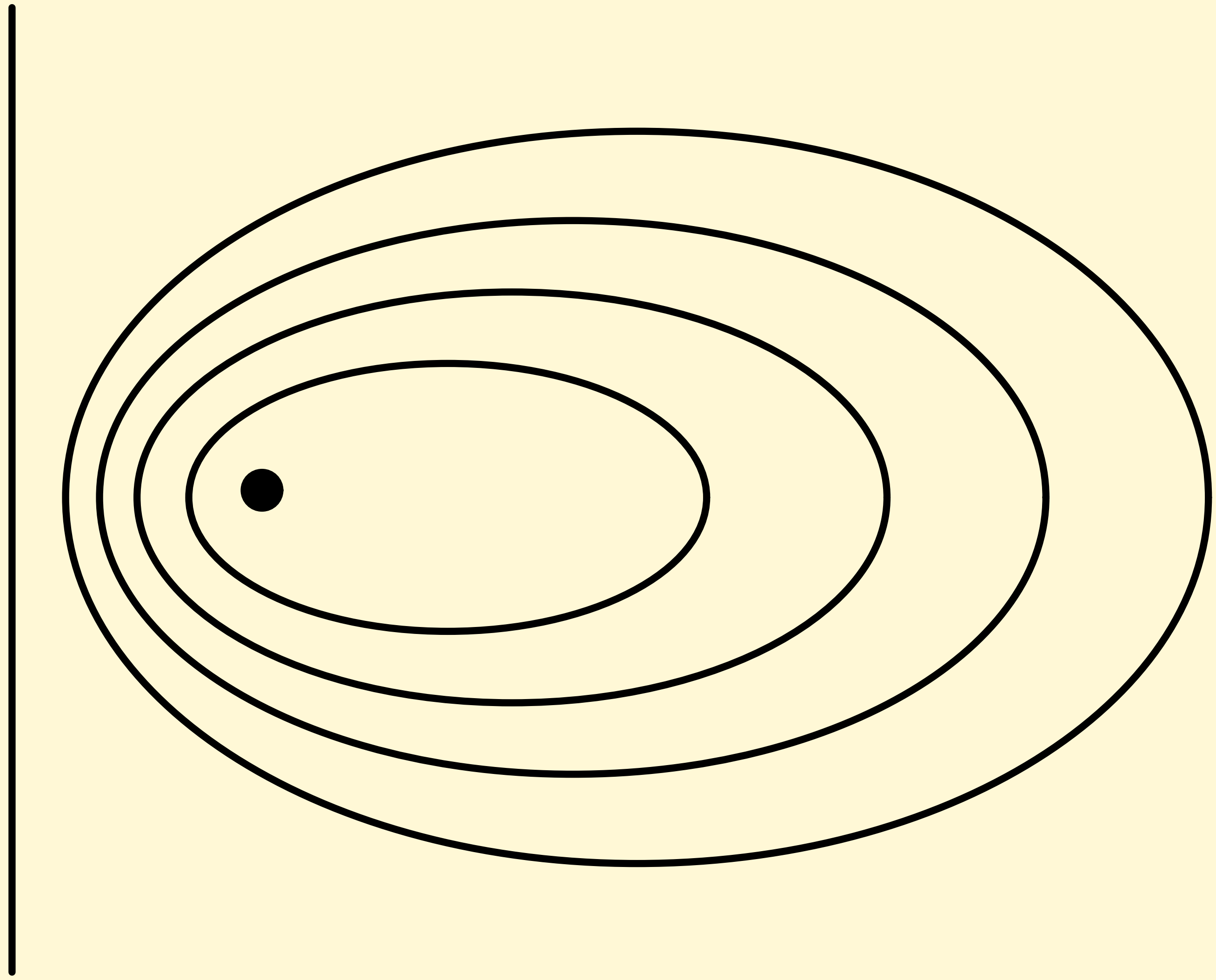

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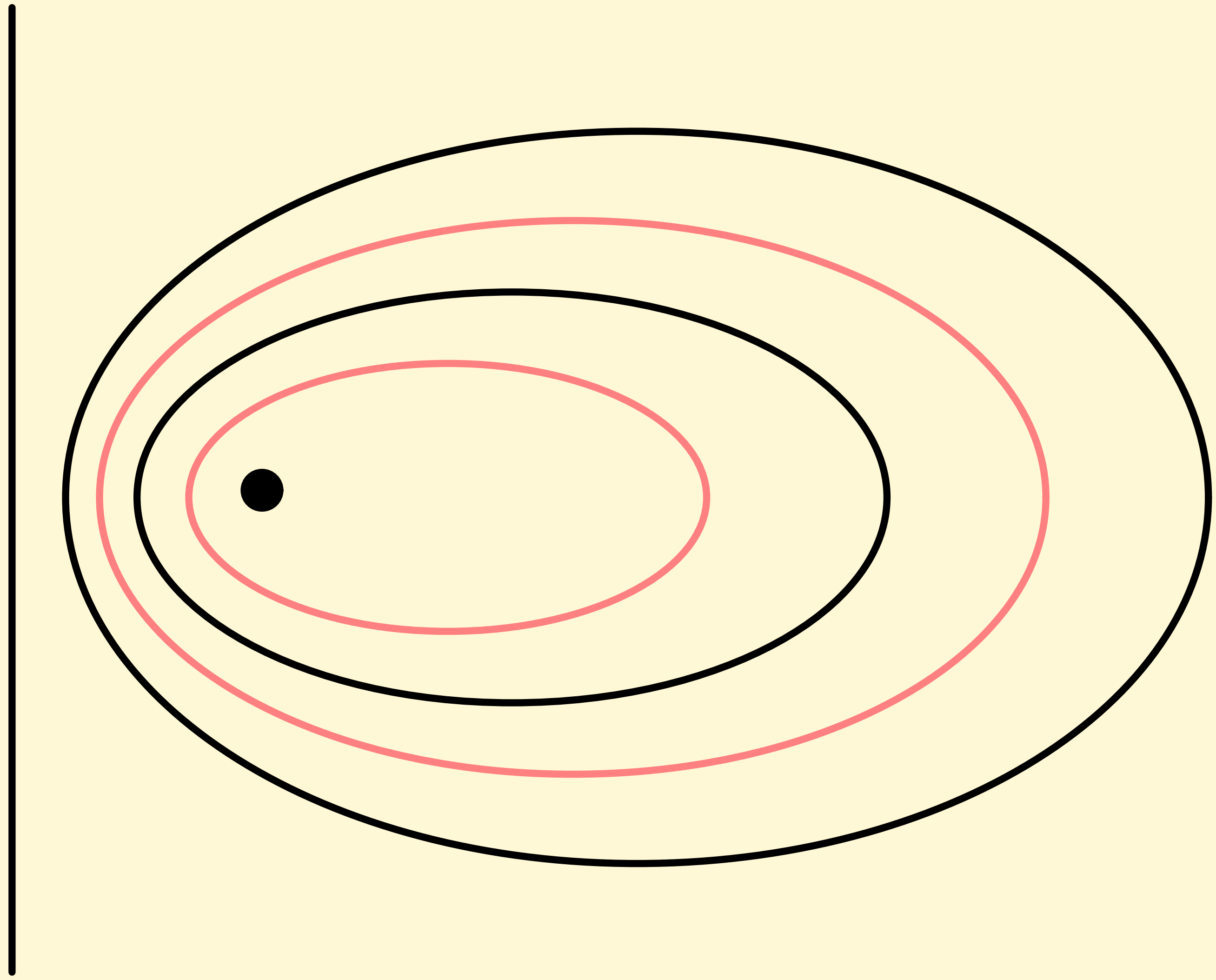
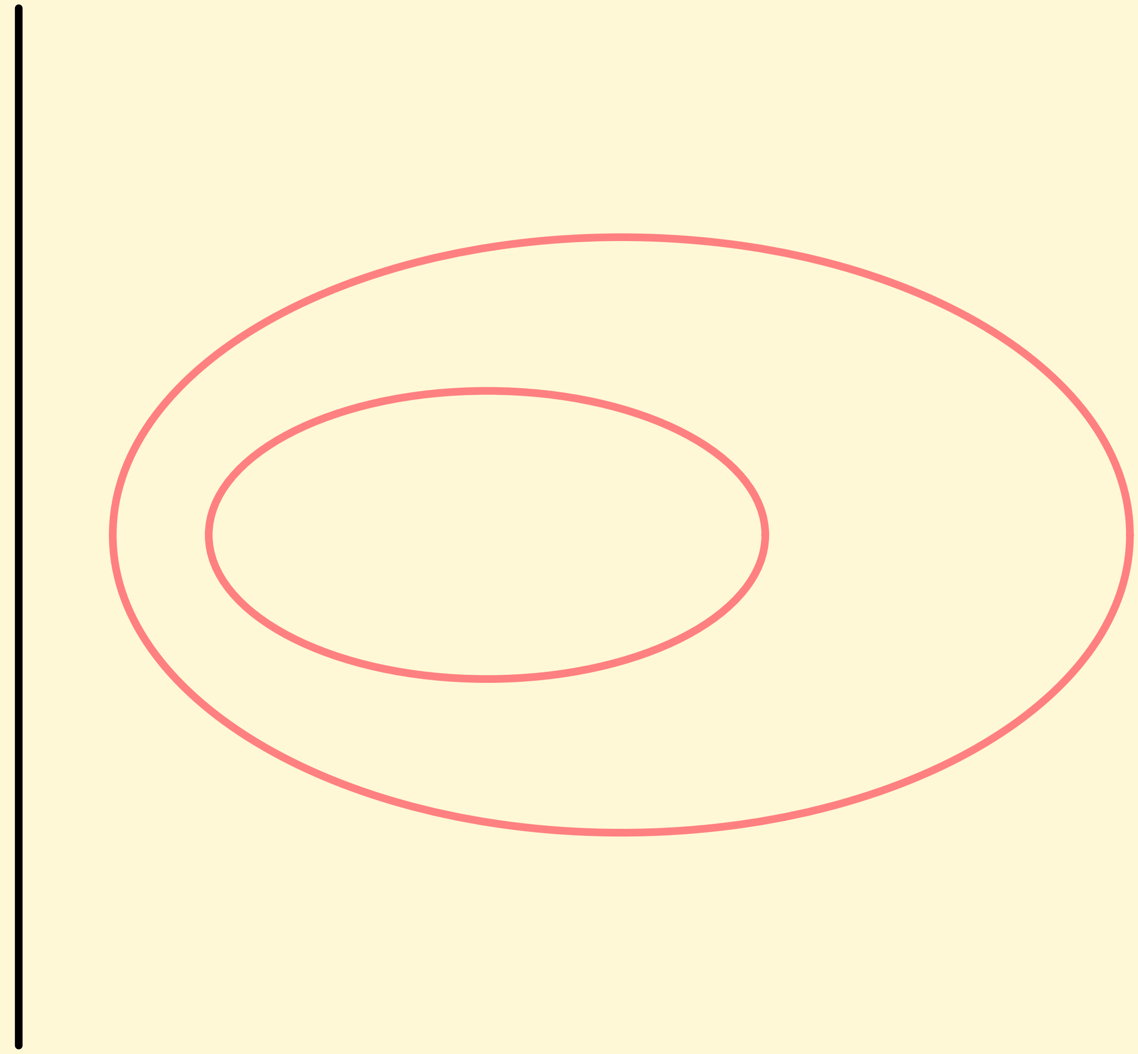


pair of dual half-conics



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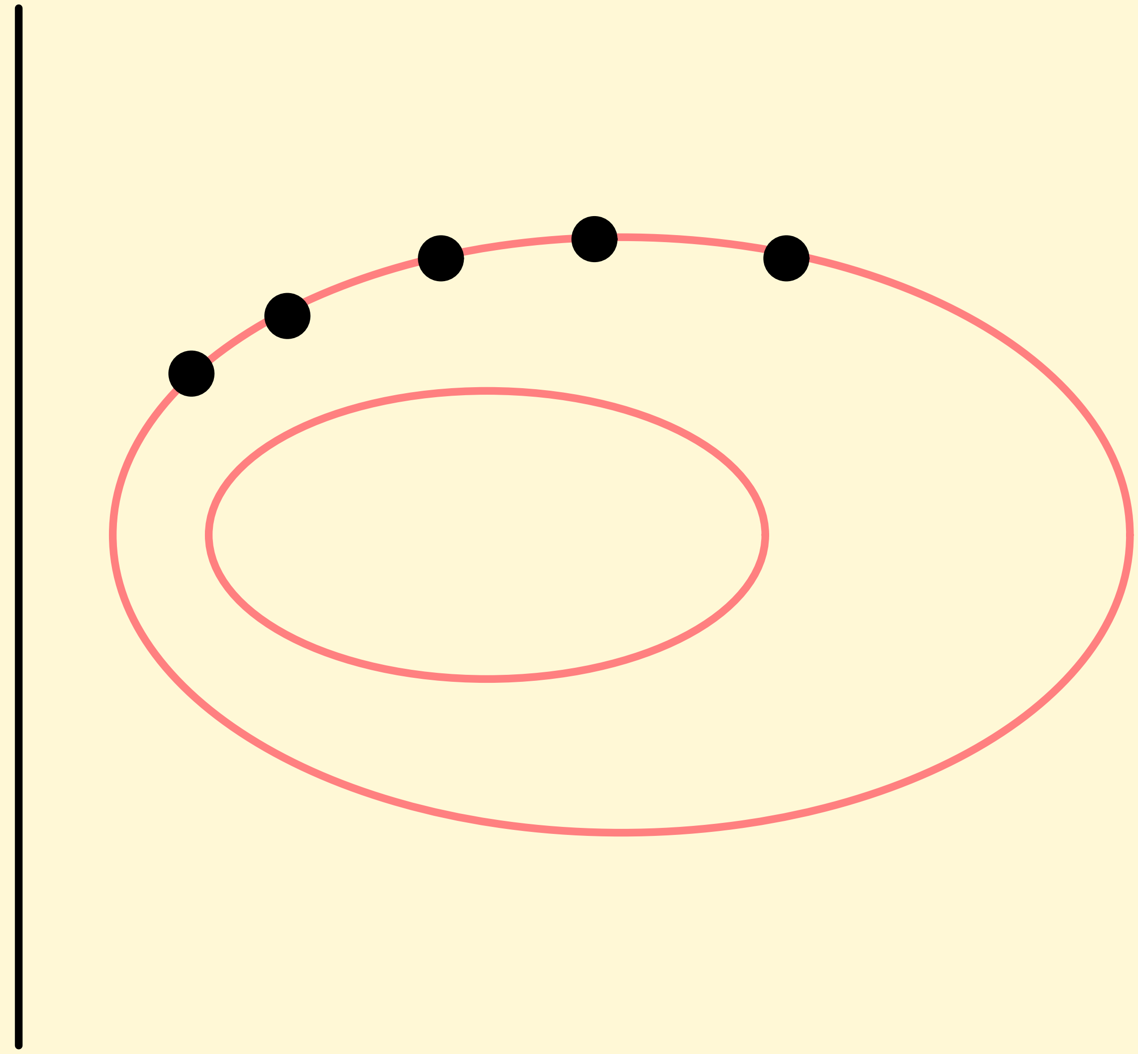

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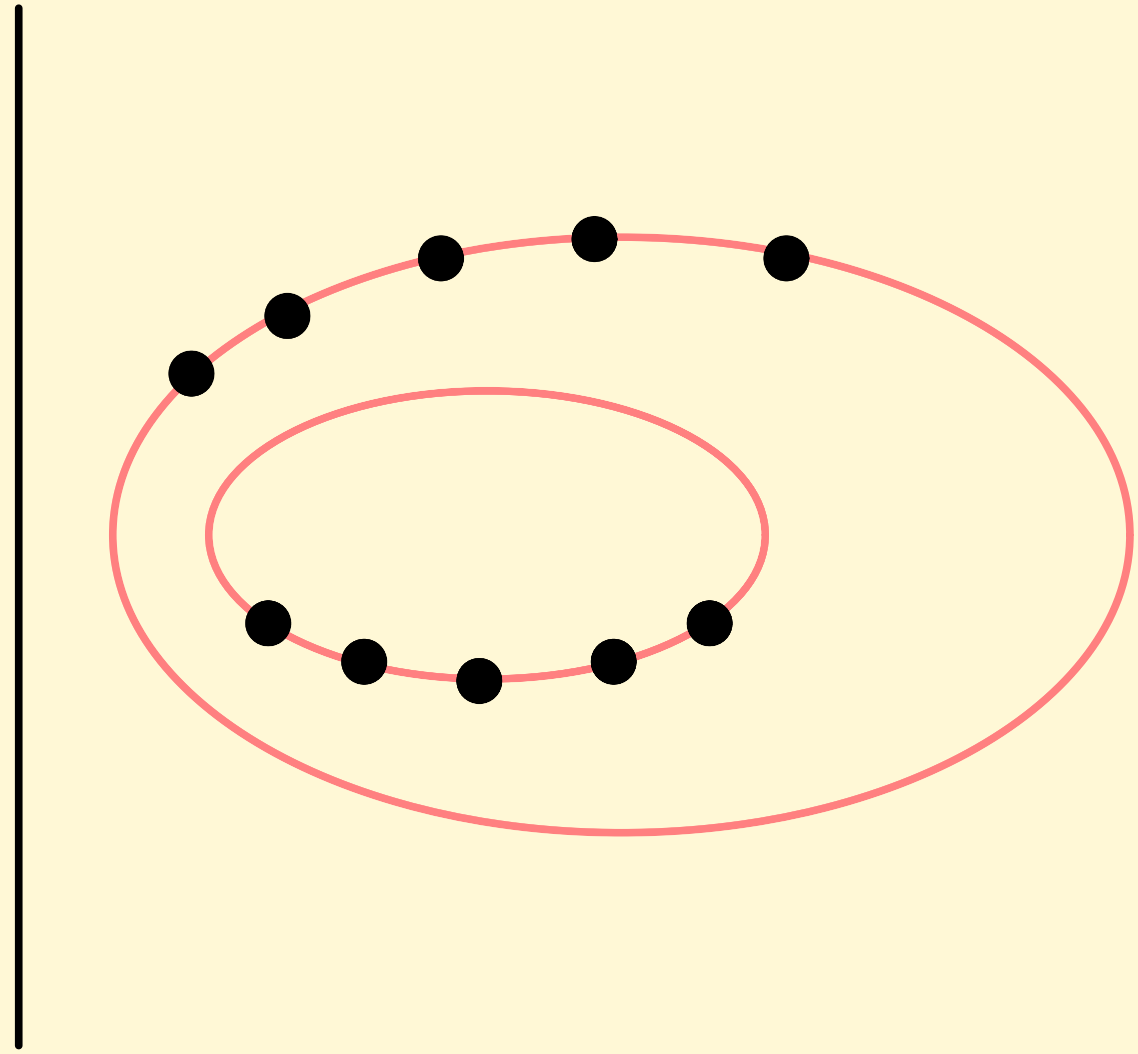

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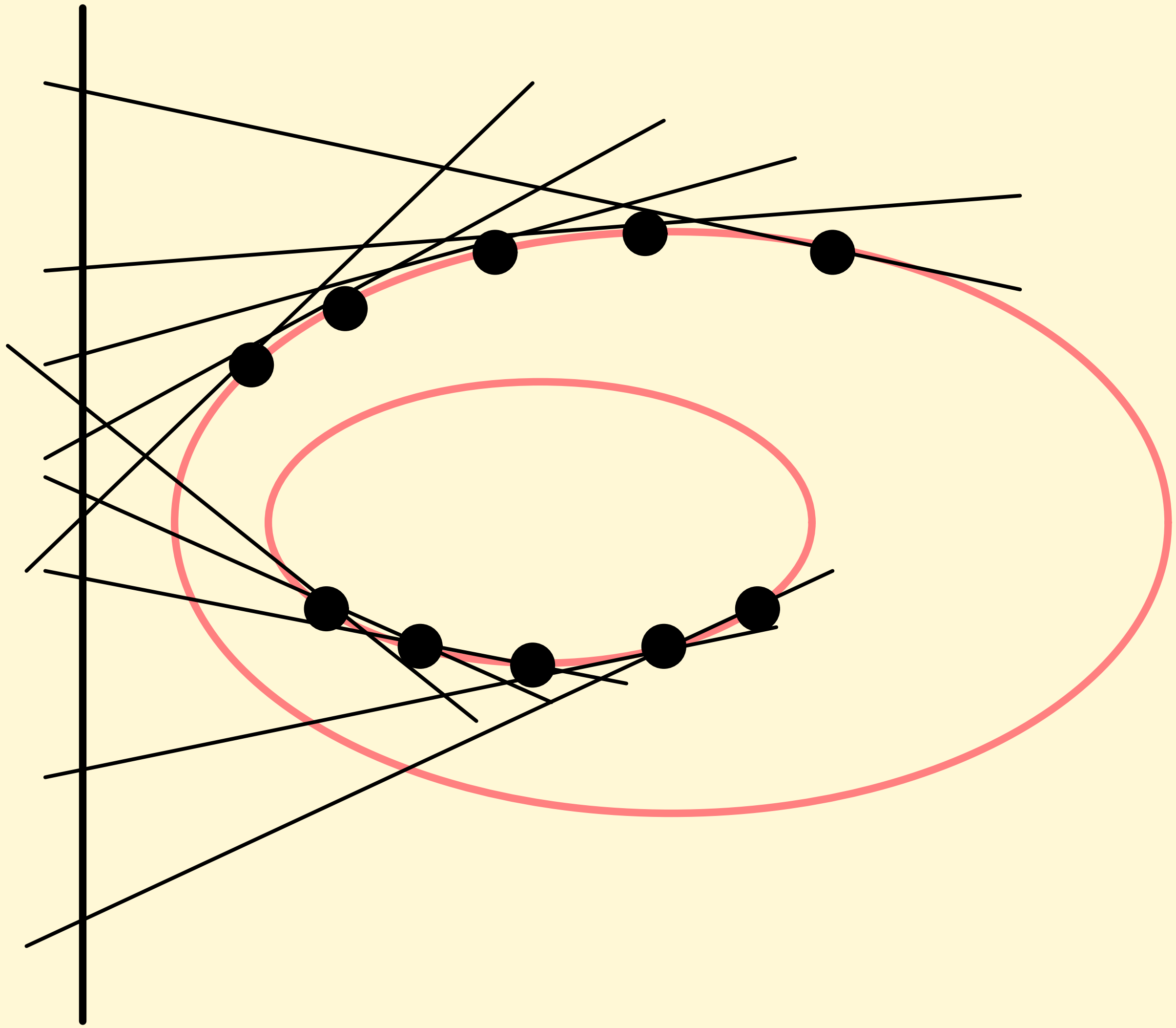

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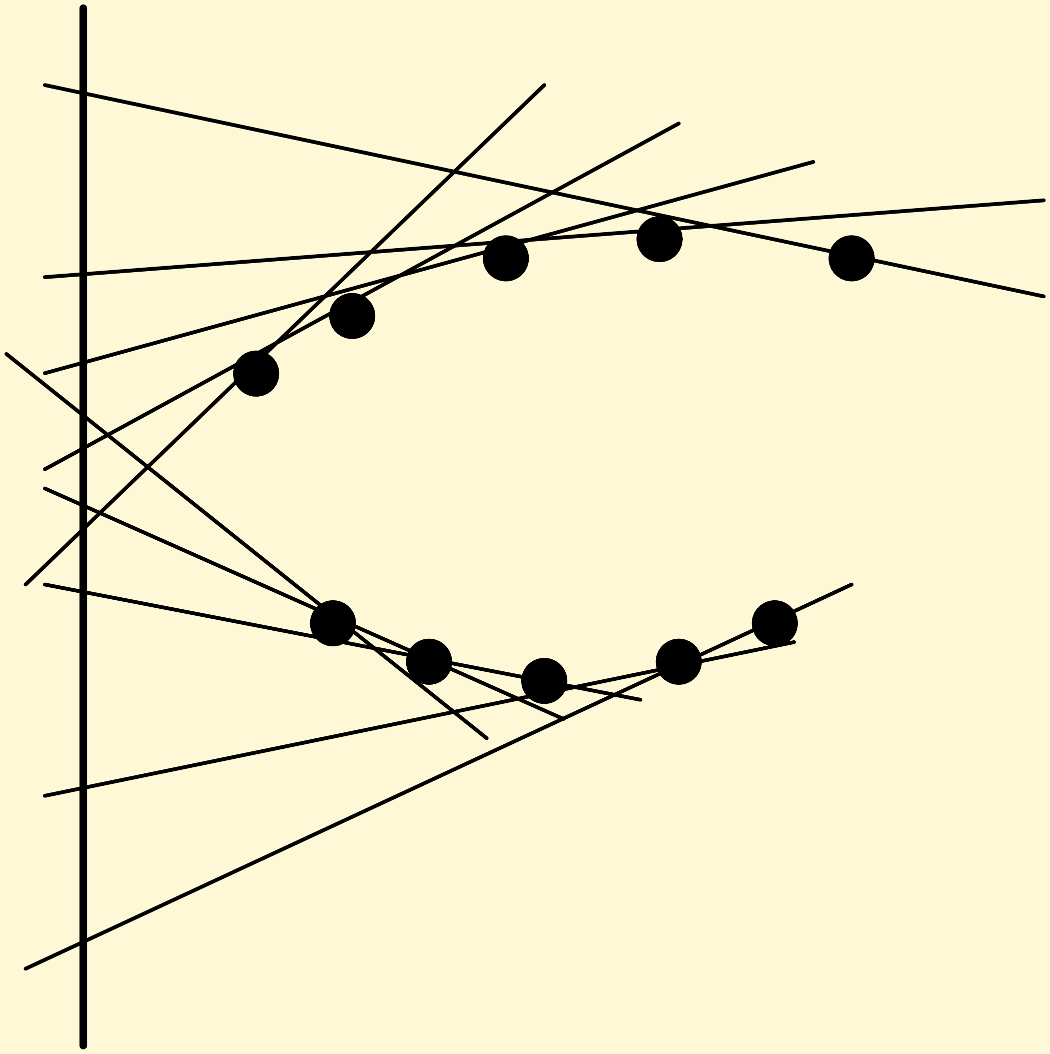

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Theorem: Let $q \equiv 1 \pmod 4$ and let C_1 and C_μ, with μ nonsquare and $\mu - 1$ a nonzero square in $GF(q)$, be conics of the pencil $\mathcal{P}$ in $PG(2,q)$. Let D_1 and D_μ be dual half-conics to C_1 and C_μ, respectively. Then the set

$$\mathcal{A} = \left(\bigcup_{m \in D_1 \cup D_\mu} m \right) \setminus \ell_\infty$$

is a minimal Kakeya set.

Theorem: Let $\mathcal{A}$ be the minimal Kakeya set described above. Then $|\mathcal{A}| = \frac{1}{8}(5q-1)(q+1)$.

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(When $q = 5$, the size is $\frac{1}{8}(24)(6) = 18$.)

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Theorem: Let $\mathcal{A}$ be the minimal Kakeya set described above. Then $|\mathcal{A}| = \frac{1}{8}(5q-1)(q+1)$.

(When $q = 5$, the size is $\frac{1}{8}(24)(6) = 18$.)

Jeremy M. Dover, Keith E. Mellinger and Kelly E. Scott, **Minimal Kakeya Sets**, to appear in *J. Comb. Des.*

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